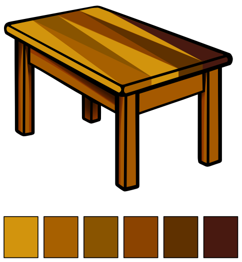
PERMUTATIONS

GAIM Newsletter Activity for **Vol. 1 | Issue 3**



If Lina wants to create a table top using planks of wood, and she has only 6 different planks to work with, how many different ways can she arrange the planks? What if she finds 10 more planks of each type of wood?

In today's activity, we are going to learn how to calculate how many different ways we can arrange the items so that we don't have to list all of the possibilities!

BEFORE WE BEGIN

In order to understand permutations, we'll first go over factorials. You're dealing with a factorial whenever you find an exclamation point (**!**) in a formula. That exclamation point means that you will multiply that number by every positive integer less than itself.

5! = 5 x 4 x 3 x 2 x 1 so **5! = 120**

You may notice that the following are all equal:

5! = 120
5 x 4! = 120
5 x 4 x 3! = 120

Lina's Folding Chair. Lina was a renowned architect, illustrator, and furniture designer in Brazil. She loved for her work to blend naturally into its surroundings, so she often used the native woods of Brazil in her furniture work.

The chair to the left is just one of her many designs!

...and so on. You could keep going all the way to 1. This helps us to create **a new rule**:

The factorial of any number is equal to that number times the factorial of itself minus one.

But here comes the funky part... what is the factorial of 0? (What is '0!'?)

The factorial of zero is actually 1. Can you figure out why?

<u>**Hint**</u>: Try to reverse engineer the example above. If you don't get it, that's okay! We'll learn more about it in the bonus section at the end.

GOAL:

To develop formulas that allow us to efficiently count arrangements of objects without having to list out all possibilities.

Remember, when dealing with permutations, position is important.

How fast does n! grow?

Well, **10!** = **3,628,800,** the exact number of seconds in six weeks. **11!** is greater than the number of seconds in a year. **12!** is greater than the number of seconds in 12 years. This might seem like a big number, until you realize that **13!** is greater than the number of seconds in a century.

Permutations WITH REPETITION

Suppose you have an ice cream cone for dessert every night for a week and there are 4 different flavors to choose from. You have 4 choices for Monday's flavor, 4 choices for Tuesday's flavor, 4 choices for Wednesday's flavor, ...and so on through Sunday.

So in total you have:

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^7$$

$$4^7 = 16{,}384$$

different arrangements to choose.

Example 1

Suppose we want to count how many passwords of length 8 we can create from the 26 letters of the alphabet. We have 26 choices for the first letter of the password, 26 choices for the 2nd letter, and so on. Therefore, there are...

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^8$$

...possible passwords that can be created.

TRY IT YOURSELF!

Instructions: Based on your work above you are ready to answer the following question.

How many ways can Lina arrange **6 types** of wood in **8 different spots** if repetition is allowed, that is, if we can use each type of wood more than once?

$$\overbrace{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6}^{8} = 6^8$$

$$6^8 = 1{,}679{,}616$$

FORMULA

*How many ways can we arrange **n** distinct objects **r** times if repetition is allowed, that is if we can use an object more than once?*

$$\overbrace{n \bullet n \bullet \ldots \bullet n}^{r \text{ times}} = n^r$$

WHY? Because we have **n** choices for the 1st plank, **n** choices for the 2nd plank, ..., **n** choices for the **r** th object.

Permutations
WITHOUT REPETITION

One way to count how many ways we can **arrange** or **permute** an entire collection of **n** objects *without* repetition is to list out all possibilities. For instance,

> *Answers are at the end of the activity in the 'Answers' Section.*

How many ways can we permute/arrange the letters AB?

There are 2 ways: **AB and BA.** - *We have 2 choices for which letter to use first and then only one choice remains for the second letter place. So,* $\mathbf{2 \cdot 1 = 2! = 2}$ ***choices***.

How many ways can we permute/arrange the letters ABC? (Answer #1.)

There are 24 ways to permute/arrange the letters ABCD. Can you list them all?

So, what happens when you want to arrange more letters? There are 3,628,800 ways to arrange the 10 letters ABCDEFGHIJ – so you wouldn't want to try listing out all possible ways to arrange those 10 letters. That is why we need to develop a formula.

Example 2

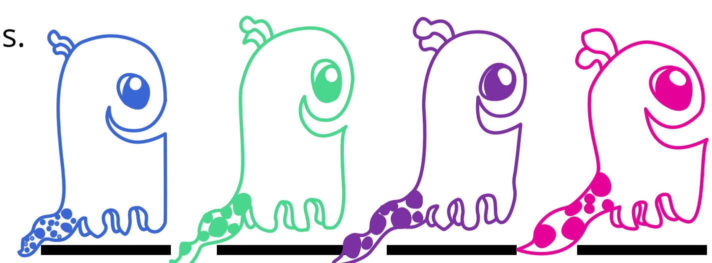
We saw that there were **24** ways to permute four letters of the alphabet. Let's see if we can determine how we arrived at 24 for our answer.

To the right is a picture of 4 different Wooshes. Suppose we want to put them all in a line.

How many choices do we have for the Woosh that is to be first in line?

Once that Woosh is selected to be first in line, how many Wooshes do we have left to choose from to be second in line?

That leaves 2 Wooshes to pick from to be 3rd in line and only one Woosh remains after that.



1	2	3	4
^	^	^	^
4 options	3 options	2 options	1 option

$$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$$

**The Wooshes above are a contribution by Rowyn Holder of Terre Haute, Indiana.*

TRY IT YOURSELF!

Let's determine the number of permutations of the five letters **ABCDE** by asking how many ways we can place a letter in the following 5 positions.

___	___	___	___	___
1	**2**	**3**	**4**	**5**

How many choices do we have for a letter to place in position 1? Once that letter is selected we have 4 letters left. How many choices do we have for a letter to select to place in position 2? As a hint, we will have 3 letters left for position 3, and then 2 letters left for position 4, and 1 letter to place in position 5. This means we have

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$$

ways to permute the five letters **ABCDE**.

FORMULA

How many ways can we arrange/permute the n distinct objects of a collection?

$$n! = n \bullet (n - 1) \bullet (n - 2) \bullet \cdots \bullet 3 \bullet 2 \bullet 1$$

WHY?

Because we have **n** ways to select an object to place first, we have **n - 1** ways to select the 2nd object to place, and **n - 2** ways to select the third object to place, and so forth until all **n** objects have been arranged.

What if we only want to arrange some of the objects in our collection? We can use the same principle.

Example 3

Suppose **3** Wooshes are standing in line to order snow cones. How many ways can we arrange those Wooshes in a line to order their snow cones? **(Answer #2)**

But, suppose we have **9** Wooshes in line for a snow cone. How many ways can we select a Woosh to order 1st, a Woosh to order 2nd, and a Woosh to order 3rd?

Since no Woosh can order more than once, we have 9 choices for who orders 1st, then 8 choices for who orders 2nd, then 7 choices for who orders 3rd. So, $9 \cdot 8 \cdot 7 = 504$ ways.

There is another clever way to write this formula. What happens if we took the number of ways to arrange all 9 Wooshes in line and divided that by the number of ways to arrange the last 6 Wooshes in line? Then we have

$$\frac{\text{\# ways to arrange 9 Wooshes}}{\text{\# ways to arrange 6 Wooshes}} = \frac{9!}{6!} = \frac{9 \cdot 8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 9 \cdot 8 \cdot 7 = 504.$$

Example 4

Suppose we want to build up an ice cream cone that always consists of **4** distinct flavors from a collection of **12** possible flavors. Then we have **12!** ways we can arrange our flavors and **8!** ways to arrange the flavors we don't want on our ice cream. This leaves us with:

$$\frac{12!}{8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 12 \cdot 11 \cdot 10 \cdot 9 = 11{,}880.$$

ways to build our ice cream cone

FORMULA

Based on our work above we are ready to answer the following question:

How many ways can we permute **r** < **n** objects from a collection of n distinct objects?

$$P(n,r) = \frac{n!}{(n-r)!}$$

Or if we have a collection of n distinct objects and we want to arrange r<n of these objects then there are

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-(r-1))$$

ways to do so.

CHALLENGE PROBLEM

In a local race, there are **eight** different sized trophies that will be awarded to the first eight runners to cross the finish line.

If 20 people enter a race, in how many ways will it be possible to award the trophies? (Answer #CP1).

If Rowyn and Meredith are two participants in the race, in how many ways can the trophies be awarded with these two runners among the top three? (Answer #CP2).

BONUS SECTION: 0 Factorial, or 0!

0! = 1, why? Let's reverse engineer 4! to get our answer.

Based on what we've learned today, 4! = 24.

We can also write this as 4 x 3! = 24. If we reverse engineer this, dividing 24 by 4, we get 3!, which is 6. We can do this over and over until we get to 0!, which equals one.

$$4! = \frac{24}{4} \quad 3! = \frac{6}{3} \quad 2! = \frac{2}{2} \quad 1! = \frac{1}{1} \quad 0! = 1$$

You can also simply think about what 0! actually measures. How many ways can you arrange nothing? Only one!

ACTIVITY SOLUTIONS

1. **6** = **3 • 2 • 1** = **3!**
2. **3 • 2 • 1** = **3!** = **6**

CP1. $P(20, 8) = \frac{20!}{(20 - 8)!}$ = **20 • 19 • 18 • 17 • 16 • 15 • 14 • 13** = **5,079,110,400 ways.**

CP2. There are **3!** ways to place Rowyn and Meredith in the first **3** positions and **p(18,6)** ways to select the remaining **6** winners. **3! • 18 • 17 • 16 • 15 • 14 • 13** = **80,196,480 ways.**

Permutations

RAPID PRACTICE

Place a checkmark in the box for the answers to the following questions:
Questions 1-4: Which of the following is equivalent?
Questions 5-8: How many ways can you arrange the letters if repetition is allowed?
Questions 9-12: How many ways can you arrange the letters if repetition is not allowed?

1) **4!**
- ☐ 24
- ☐ 34
- ☐ 54
- ☐ 84

2) **6!**
- ☐ 420
- ☐ 520
- ☐ 620
- ☐ 720

3) **10!**
- ☐ 10000
- ☐ 11 x 9!
- ☐ 10 x 9!
- ☐ 100 x 8!

4) **12!**
- ☐ 144
- ☐ 12 x 10!
- ☐ 132 x 10!
- ☐ 132 x 5!

5) **abc**
- ☐ 27
- ☐ 120
- ☐ 3125
- ☐ 6^6

6) **abcde**
- ☐ 27
- ☐ 120
- ☐ 3125
- ☐ 6^6

7) **aeiou**
- ☐ 27
- ☐ 120
- ☐ 3125
- ☐ 6^6

8) **lmnopq**
- ☐ 27
- ☐ 120
- ☐ 3125
- ☐ 6^6

9) **Mary**
- ☐ 24
- ☐ 7!
- ☐ 20
- ☐ $\frac{8!}{(2!\ 2!)}$

10) **Charlie**
- ☐ 24
- ☐ 7!
- ☐ 20
- ☐ $\frac{8!}{(2!\ 2!)}$

11) **Bobby**
- ☐ 24
- ☐ 7!
- ☐ 20
- ☐ $\frac{8!}{(2!\ 2!)}$

12) **Isabella**
- ☐ 24
- ☐ 7!
- ☐ 20
- ☐ $\frac{8!}{(2!\ 2!)}$

TRY IT YOURSELF!

You can practice permutations with any number you can dream! How many ways can you arrange the letters in your name with and without repetition?

Permutations

WORD PROBLEMS

Complete the following word problems. If the words are confusing you, focus first on the numbers. You may find it easier to check all of the numbers' divisibility first!

Problem #1

In Coach Bee's PE class, she has 25 students. They are competing in a mock 2020 olympics:

WP1-a. How many possible ways can the class be ranked, if they ranked them fully between 1-25 assuming there are no ties? Please answer in the form of a factorial.

WP1-b. Assuming there are no ties, how many potential ways can a Gold, Silver and Bronze be awarded in Coach Bee's classroom?

WP1-c. Assuming no ties, and 3 medals (Gold, Silver, Bronze), how many different ways can the three ***medals*** be awarded in Coach Bee's classroom? The type of medal won does not matter.

Problem #2

WP2. Coach Bee has 7 players that have joined the basketball team. She wants to start a New Team of 5 players at every basketball game. The season is 10 weeks long and they have 2 games every week. Would this be possible and why?

Problem #3

Mrs. Harrada's child David wants to make sure he has enough school outfits without wearing an outfit twice in any given month. David has 4 pairs of pants and 3 different shirts. Assume the following: 20 school days in a month. Individual clothing is cleaned after each use.

WP3-a. Does he have enough clothing to last a whole month without wearing the same outfit twice?

WP3-b. He goes to ask his mom about getting some additional clothes. HIs mom agrees to buy two more articles of clothing. What potential combinations of pants/shirts can he buy in order to have a sufficient number of outfits?

RAPID PRACTICE ANSWERS

1) **4!**

- [x] 24
- [ ] 34
- [ ] 54
- [ ] 84

2) **6!**

- [ ] 420
- [ ] 520
- [ ] 620
- [x] 720

3) **10!**

- [ ] 10000
- [ ] 11 x 9!
- [x] 10 x 9!
- [ ] 100 x 8!

4) **12!**

- [ ] 144
- [ ] 12 x 10!
- [x] 132 x 10!
- [ ] 132 x 5!

5) **abc**

- [x] 27
- [ ] 120
- [ ] 3125
- [ ] 6^6

6) **abcde**

- [ ] 27
- [ ] 120
- [x] 3125
- [ ] 6^6

7) **aeiou**

- [ ] 27
- [ ] 120
- [x] 3125
- [ ] 6^6

8) **lmnopq**

- [ ] 27
- [ ] 120
- [ ] 3125
- [x] 6^6

9) **Mary**

- [x] 24
- [ ] 7!
- [ ] 20
- [ ] $\frac{8!}{(2!\ 2!)}$

10) **Charlie**

- [ ] 24
- [x] 7!
- [ ] 20
- [ ] $\frac{8!}{(2!\ 2!)}$

11) **bobby**

- [ ] 24
- [ ] 7!
- [x] 20
- [ ] $\frac{8!}{(2!\ 2!)}$

12) **Margaret**

- [ ] 24
- [ ] 7!
- [ ] 20
- [x] $\frac{8!}{(2!\ 2!)}$

WORD PROBLEM ANSWERS

WP1-a. 25!

WP1-b. $25 \cdot 24 \cdot 23 = 13800$

WP1-c. $\frac{25 \cdot 24 \cdot 23}{3!} = 2300$

WP2. Yes, it would be possible. To get the number of possible teams, we need to choose **5 students** and note that the order of choice doesn't matter! So we get $(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)/5! =$ **21 different teams**, which is more than 20.

WP3-a. No, he does not have enough clothing.

WP3-b. He can buy either **1 shirt and 1 pair of pants**, or he can buy **2 shirts**. He cannot buy 2 pairs of pants.