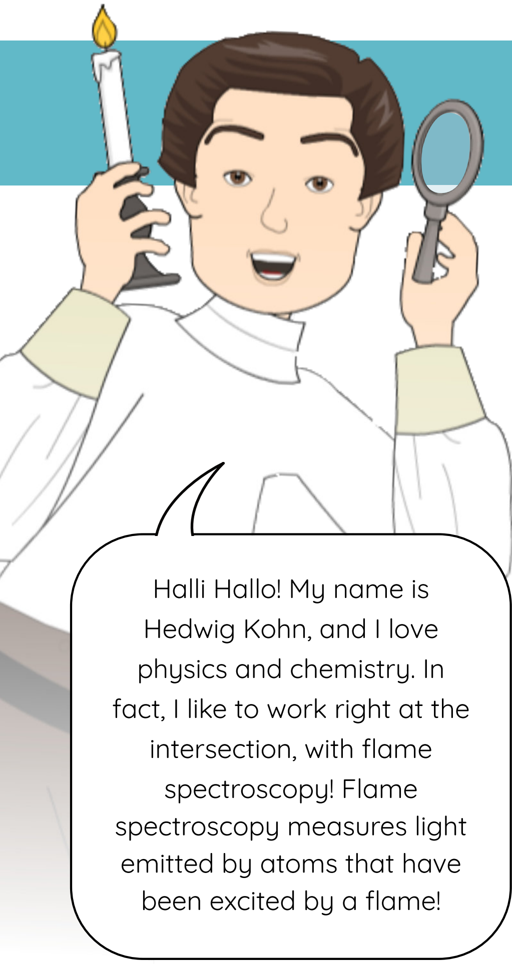
COMBINATIONS
WITHOUT REPETITION



GAIM Newsletter Activity for **Vol. 1 | Issue 4**

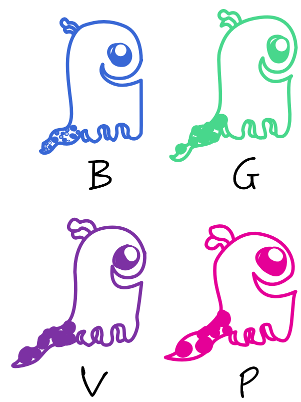
In Newsletter 3 we learned how to count permutations. Remember that a **permutation** is an arrangement of objects, so order matters. But, order doesn't matter when dealing with combinations. You can think of a **combination** as an unordered permutation or vice-versa-- a permutation is an ordered combination.

Remember, the formula for computing the number of ways we can permute **r** objects from a collection of **n** objects is:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example 1

Consider the 4 colored Wooshes to the right. Let's list out all possible permutations of the Wooshes and the combinations of the Wooshes.



ORDER DOES MATTER:

B G V P	B P V G	P G B V	G B P V	G V B P	V B P G
B G P V	B P G V	P G V B	G B V P	G V P B	V B G P
B V G P	P V G B	P B V G	G P B V	V G P B	V P G B
B V P G	P V B G	P B G V	G P V B	V G B P	V P B G

ORDER DOES NOT MATTER: BGVP

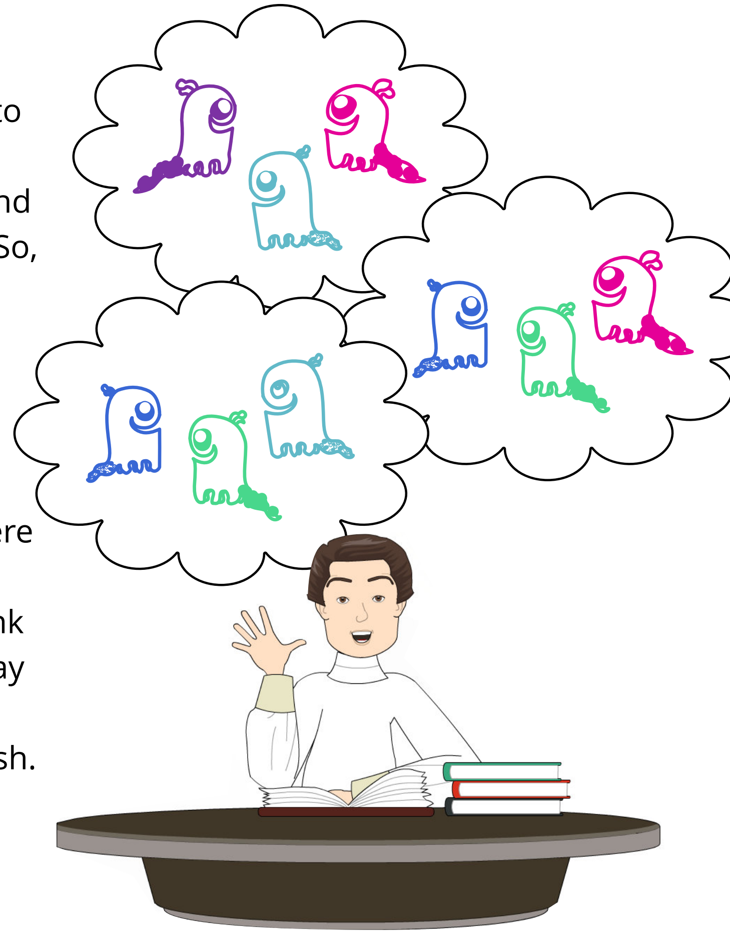
So, when order matters there are

$$\mathbf{P(4, 4) = 4! = 24 \text{ ways}}$$

24 ways we can permute or arrange the 4 Wooshes.
But, when order doesn't matter, there is only 1 collection or combination of 4 Wooshes.

The key to counting combinations is to count the permutations and then to remove any instances where you may have counted the same combination more than once (since order doesn't matter). You can achieve this by dividing the total by the number of ways to order your selected objects. ***If that was confusing, don't worry! Let's try an example.***

Example 2



Suppose there are 5 Wooshes and Hedwig wants to select three of them to study with her. There are 5 choices for the first Woosh, 4 choices for the second Woosh, and 3 for the final Woosh to be selected. So, we might think the answer is

$$5 \cdot 4 \cdot 3 = \frac{5!}{(5-3)!} = P(5, 3) = 60$$

But, we would be overcounting if we said there were 60 ways. Why? One way to select three Wooshes would be to first pick the blue Woosh, then the pink one, and finally the green Woosh. But, another way to pick three Wooshes is to select the pink Woosh first, then the green one, and finally the blue Woosh. So, we counted the same collection of Wooshes {blue, pink, green} twice because they are the same when order doesn't matter.

And that isn't the only way to select the blue, pink, and green Woosh. Can you find how many other ways Hedwig could have selected those three Wooshes?

We just want to know how many unique combinations of Wooshes Hedwig could select. How do we fix this overcounting issue? We take the number of ways to select three Wooshes from the 5 Wooshes and divide by the number of ways we can order the 3 Wooshes – this gets rid of the extras!

# of ways to select 3 from 5

$$\binom{5}{3} = \frac{P(5, 3)}{3!} = \frac{5!}{(5-3)! \cdot 3!} = \frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \frac{5 \cdot 4}{2!} = \frac{20}{2} = 10$$

# of ways to order

This number is small enough that you could check by listing out all possibilities.

TRY IT YOURSELF!

Instructions: Try listing out all of the possibilities and check your answer against **A** in the 'Solutions' section.

Let's try another problem before we officially develop the formula.

Example 3

How many ways can we select four scoops of ice cream to place in our bowl, without repeating any flavors, from a selection of 10 different ice cream flavors? Notice that order isn't important since we just want to select the flavors.

We have $\mathbf{10 \cdot 9 \cdot 8 \cdot 7 = P\,(10, 4) = \dfrac{10!}{(10 - 4)!} = 5040}$

ways to pick 4 different ice cream flavors.

Since order doesn't matter, we must divide the above number by the number of ways to arrange (or permute) the 4 flavors which is

$$\mathbf{P\,(4, 4) = 4! = 24}$$

This gives us

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = \frac{10 \cdot \cancel{9} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = \frac{5 \cdot 3 \cdot 2 \cdot 7}{1} = 210$$

210 ways to choose 4 flavors of ice cream.

Notice this is really $\dfrac{P(10,4)}{4!} = \dfrac{10!}{(10 - 4)! \cdot 4!}$

FORMULA

Let's see if you can develop a general formula now. If we have n objects and we want to select any k of them, without repetition, then we have

Sidenote*: This is pronounced "n choose k" and is often called a "binomial coefficient".*

$$\binom{n}{k} = C\,(n\,,\,k) = \frac{P(n, k)}{k!} = \frac{n!}{(n - k) \cdot k!}$$

You could memorize a formula or remember that you get combinations by removing the order from permutations.

TRY IT YOURSELF!

Instructions: Complete the following problems. Answers are in the 'Solutions' section.

1. Compute the following. Did you notice anything unusual about your answers?

A) $\binom{9}{3} =$

B) $\binom{9}{6} =$

C) $\binom{12}{4} =$

D) $\binom{12}{8} =$

2. If Ophelia has 13 books by her favorite author, in how many ways can she select 7 books to take on her road trip?

$\binom{\ }{\ } =$

3. Katerina has gone to the store to purchase paint for her model rocket. The store has 25 different colors for model paints and Katerina wants to use 5 distinct colors on her rocket. In how many ways can she select the 5 colors to paint her rocket?

$\binom{\ }{\ } =$

CHALLENGE PROBLEMS

A bin has the following Wooshes: 4 red, 5 blue, 3 green, and 2 purple making 14 Wooshes total. Each Woosh is wearing a jersey/shirt with a different number on it, making, for example, two red Wooshes different from one another.

1. How many groups of 4 Wooshes are possible?

2. How many groups of 4 Wooshes are there such that each one is a different color?

3. How many sets of 4 are there in which <u>at least 2</u> of the Wooshes are red?

4. How many sets of 4 are there in which none are red, but at least one is green?

CHALLENGE PROBLEM ANSWERS

A bin has the following Wooshes: 4 red, 5 blue, 3 green, and 2 purple making 14 Wooshes total. Each Woosh is wearing a jersey/shirt with a different number on it.

1. How many groups of 4 Wooshes are possible?

Since all the Wooshes have on distinct jerseys, we have C(14,4) =1001.

2. How many groups of 4 Wooshes are there such that each one is a different color?

C(4,1) x C(5,1) x C(3,1) x C(2,1) = 4x5x3x2 = 120 because we are choosing one Woosh of each color.

3. How many sets of 4 are there in which at least 2 of the Wooshes are red?

We could have 2 red Wooshes in our set, or 3 red Wooshes in our collection, or 4 red Wooshes in our collection. It doesn't matter what other colors are in our collection but we must make sure we have 4 Wooshes in our collection.

A. There are C(4,2) ways to pick 2 red Wooshes and C(10,2) ways to select 2 Wooshes that are not red. This gives C(4,2)xC(10,2) = 270.

B. There are C(4,3) ways to pick 3 red Wooshes and C(10,1) ways to select another Woosh to make our set of 4. This gives us C(4,3)xC(10,1) = 40.

C. There are C(4,4) ways to pick 4 red Wooshes but we don't need any other Wooshes to make our collection, so here we have C(4,4)=1 or C(4,4)xC(10,0) = 1x1 = 1.

In total we have 270+40+1 = 311.

4. How many sets of 4 are there in which none are red, but at least one is green?

We can have either 1 green & 3 other non-red Wooshes, 2 green & 2 other non-red Wooshes, or 3 green & 1 other non-red Woosh in our collection.

Also, since we don't want red Wooshes, we only have 10 Wooshes from which to pick.

In total, we have C(3,1)xC(7,3) + C(3,2)xC(7,2) + C(3,3)xC(7,1) = 3x35+ 3x21 + 1x7 = 175 sets in which there is at least one green Woosh but no red Woosh.

SOLUTIONS

A. If we use A, B, C, D, and E to represent the 5 Wooshes:

ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCD, BDE, CDE

1A. $$\binom{\mathbf{9}}{\mathbf{3}} = \frac{9!}{(9-3)!\cdot 3!} = \frac{9!}{6!\cdot 3!} = \frac{\overset{3\cdot 4\cdot 7}{\cancel{9}\cdot\cancel{8}\cdot 7\cdot\cancel{6!}}}{\cancel{6!}\,(\cancel{3}\cdot\cancel{2}\cdot 1)} = \frac{3\cdot 4\cdot 7}{1} = 84$$

1B. $$\binom{\mathbf{9}}{\mathbf{6}} = \frac{9!}{(9-6)!\cdot 6!} = \frac{9!}{3!\cdot 6!} = \frac{\overset{3\cdot 4\cdot 7}{\cancel{9}\cdot\cancel{8}\cdot 7\cdot\cancel{6!}}}{(\cancel{3}\cdot\cancel{2}\cdot 1)\,\cancel{6!}} = \frac{3\cdot 4\cdot 7}{1} = 84$$

1C. $$\binom{\mathbf{12}}{\mathbf{4}} = \frac{12!}{(12-4)!\cdot 4!} = \frac{12!}{8!\cdot 4!} = \frac{\overset{3\cdot 11\cdot 5\cdot 3}{\cancel{12}\cdot 11\cdot\cancel{10}\cdot\cancel{9}\cdot\cancel{8!}}}{\cancel{8!}\,(\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)} = \frac{3\cdot 11\cdot 5\cdot 3}{1} = 495$$

1D. $$\binom{\mathbf{12}}{\mathbf{8}} = \frac{12!}{(12-8)!\cdot 8!} = \frac{12!}{4!\cdot 8!} = \frac{\overset{3\cdot 11\cdot 5\cdot 3}{\cancel{12}\cdot 11\cdot\cancel{10}\cdot\cancel{9}\cdot\cancel{8!}}}{(\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)\,\cancel{8!}} = \frac{3\cdot 11\cdot 5\cdot 3}{1} = 495$$

2. $$\binom{13}{7} = \frac{13!}{(13-7)!\cdot 7!} = \frac{13!}{6!\cdot 7!} = \frac{\overset{13\cdot 2\cdot 11\cdot 2\cdot 3\cdot 2}{13\cdot\cancel{12}\cdot 11\cdot\cancel{10}\cdot\cancel{9}\cdot\cancel{8}\cdot\cancel{7!}}}{(\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1)\,\cancel{7!}} = \frac{13\cdot 2\cdot 11\cdot 2\cdot 3\cdot 2}{2\cdot 1} = 1716$$

3. $$\binom{25}{5} = \frac{25!}{(25-5)!\cdot 5!} = \frac{25!}{20!\cdot 5!} = \frac{\overset{5\cdot 6\cdot 23\cdot 11\cdot 7}{\cancel{25}\cdot\cancel{24}\cdot 23\cdot\cancel{22}\cdot\cancel{21}\cdot\cancel{20!}}}{\cancel{20!}\,(\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)} = \frac{5\cdot 6\cdot 23\cdot 11\cdot 7}{1} = 53130$$