COMBINATIONS
WITH REPETITION

GAIM Newsletter Activity for **Vol. 1 | Issue 5**

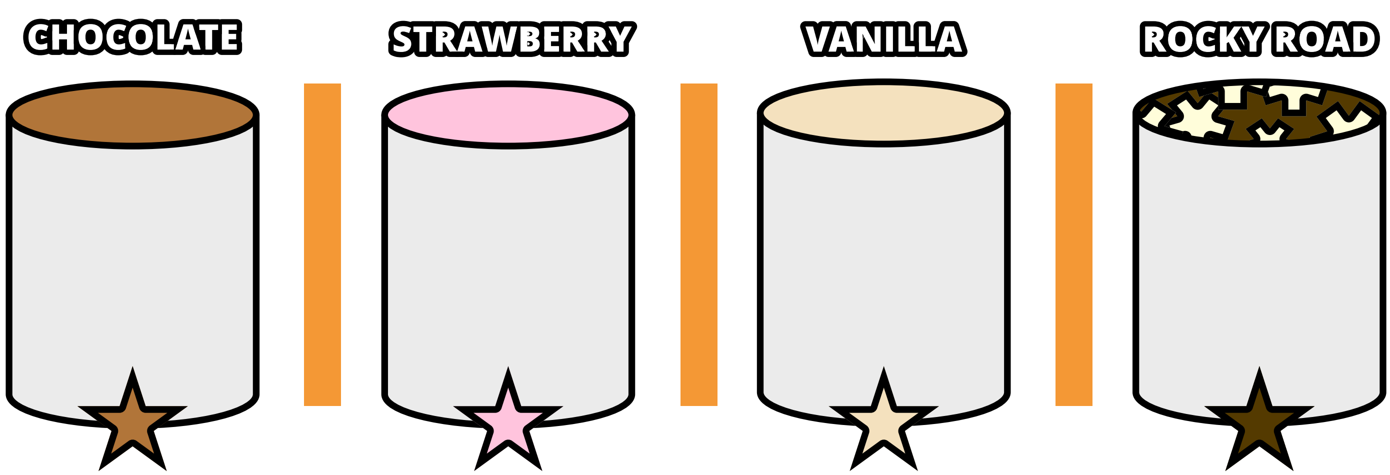
Hi girls! My name is Shirley Holmes, and I'm the face of Girls' Adventures in Math. I love going on adventures and learning new things, especially by problem solving! I see math everywhere, so it's my favorite way to explore.

In Combinations <u>without</u> Repetition, we learned how to count combinations where objects should only be counted once. Calculating combinations <u>with</u> repetition requires a bit more thought. Let's start with an example!

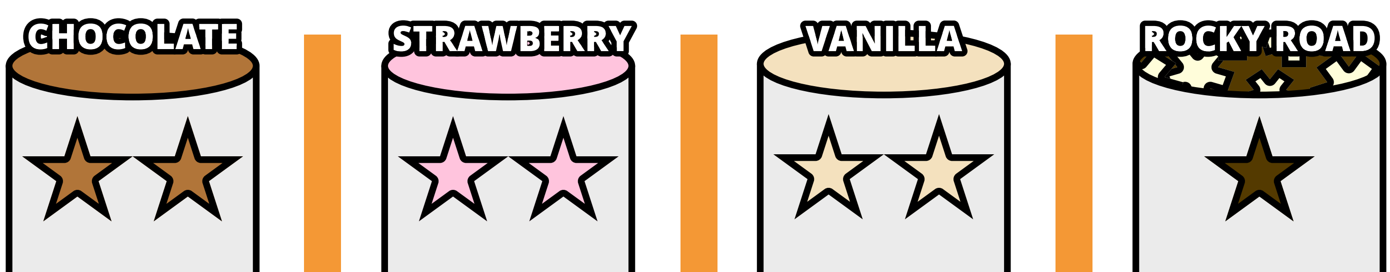
Example 1

After a soccer game, seven friends stop at an ice cream parlor serving only four flavors: chocolate, strawberry, vanilla, and rocky road. Each of the 7 girls chooses only <u>one</u> scoop of ice cream. **How many different ways can 7 people choose only one scoop out of four flavors?**

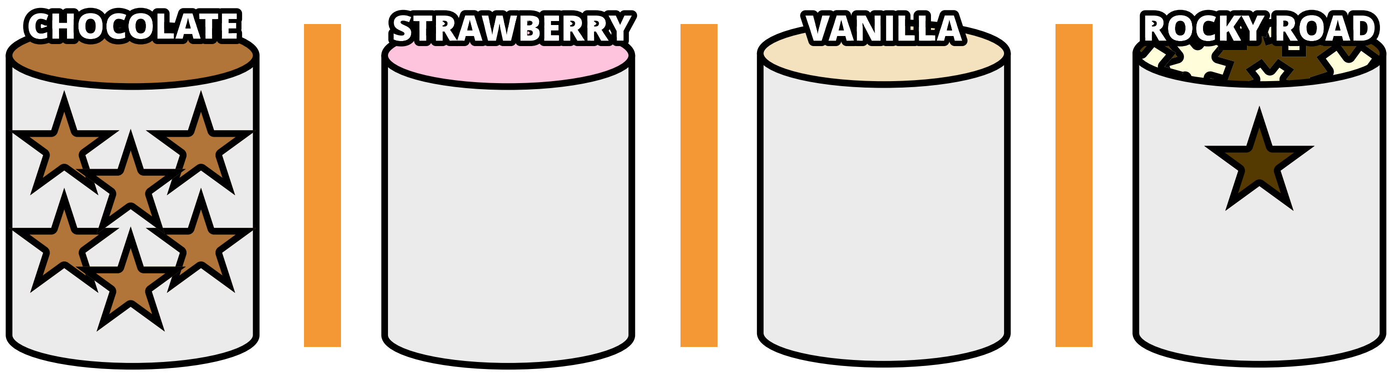
To fulfill the order, the ice cream parlor needs to know how many chocolate scoops, how many strawberry scoops, how many vanilla scoops, and how many rocky road scoops are needed. The restaurant has 4 bins labeled chocolate, strawberry, vanilla, and rocky road:



Notice each bin has a divider between it represented by an orange bar, |. Let's place a star, ★ in the appropriate bin to represent one scoop ordered of that flavor. For instance, suppose the friends order 2 chocolate scoops, 2 strawberry scoops, 2 vanilla scoops, and 1 rocky road scoop, **which we would represent as** ★★ | ★★ | ★★ | ★

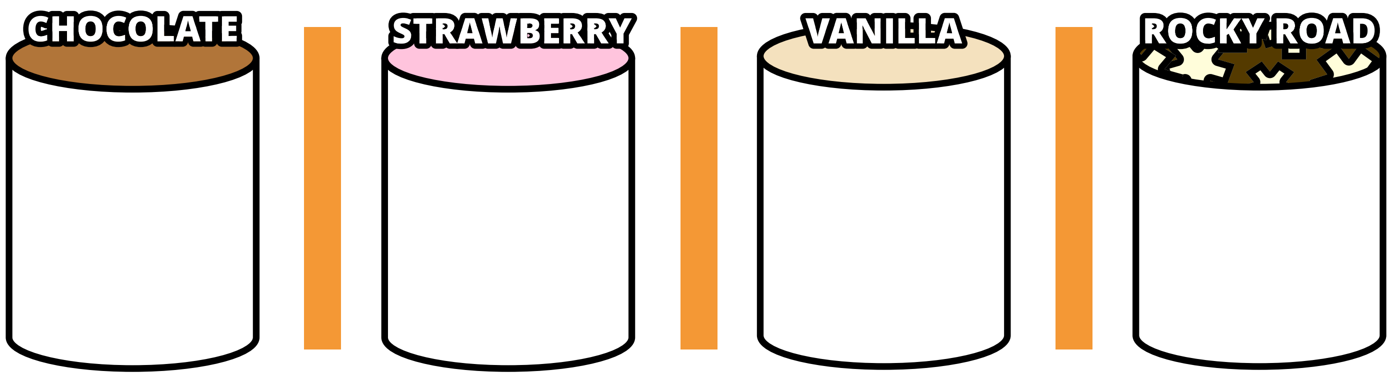


If the friends order 6 chocolate scoops and 1 rocky road scoop, **we would have the following representation:** ★★★★★★ | | | ★ and our bins would look like:



TRY IT YOURSELF!

Instructions: Place ★'s in the bins below to help you find the ★'s and |'s representation if the friends ordered **1 strawberry** scoop, **2 vanilla** scoops, and **4 rocky road** scoops.

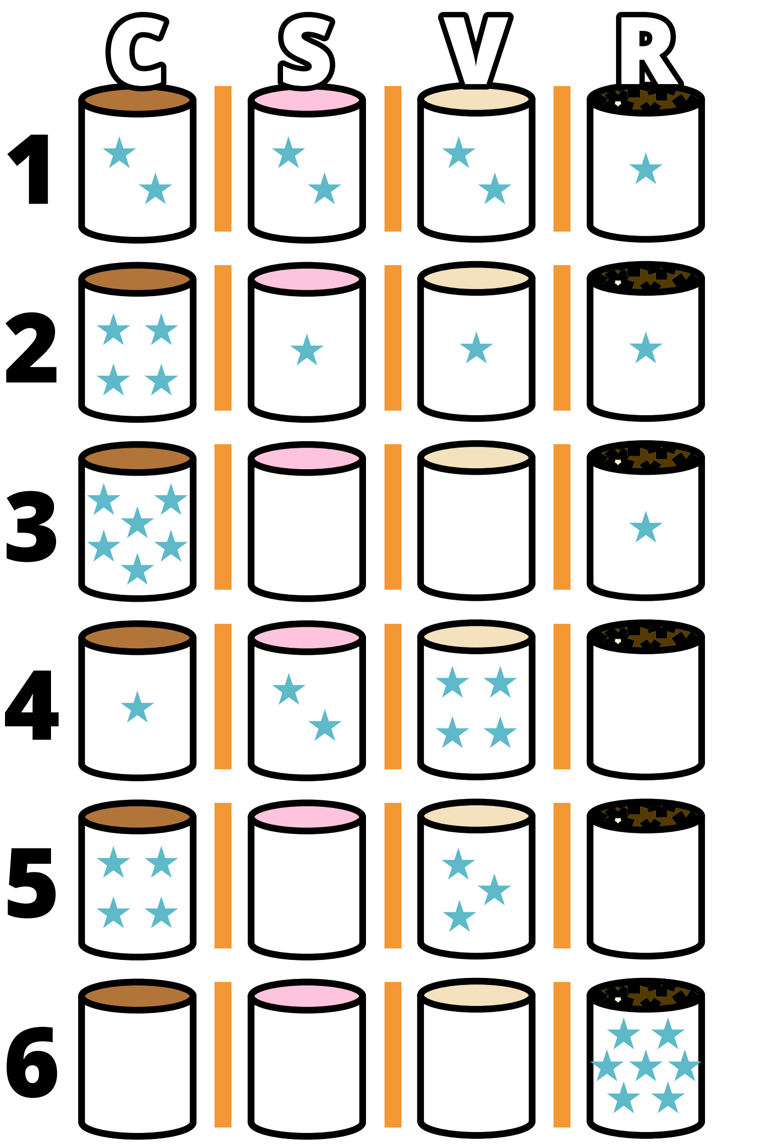


Can you rewrite the above as stars and bars?

| | |

PRACTICE ACTIVITY

In the columns below we list some possible purchases. For each set of bins in the first column, there is a corresponding |'s and ★'s representation in the second column. Can you match them? Write the number of the order from the first column in the purple box next to the corresponding bars and stars representation in the second column.



Place the numbers of the corresponding orders in the boxes below.

☐ | | | ★ ★ ★ ★ ★ ★ ★

☐ ★ ★ ★ ★ | | ★ ★ ★ |

☐ ★ ★ | ★ ★ | ★ ★ | ★

☐ ★ ★ ★ ★ | ★ | ★ | ★

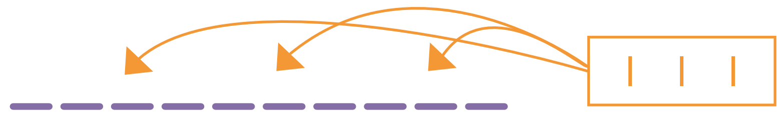
☐ ★ | ★ ★ | ★ ★ ★ ★ |

☐ ★ ★ ★ ★ ★ ★ | | | ★

Notice that each set of bars and stars occupies 10 spaces.

For each order, we see that each ★ to the left of the first bar (|) represents a **C**, each ★ between the first and second bar represents an **S**, and each ★ between the second and third bar represents a **V**, and each ★ after the third bar represents an **R**. The bars are acting as dividers between the choices of ice cream flavors and each ★ represents how many of those flavors are ordered. **For each combination, notice that each set of bars is in different spots.**

Now a relationship has been established between the two columns, where we know how to count the bars and stars in the second column. We have 10 spots in which we need to place 7 ★'s and 3 |'s. Once we place the 3 |'s, the remaining 7 spots must be ★'s. We have 10 spots, and we must pick (or <u>choose</u>) 3 of them to represent a bar (|).



In other words, we need to compute $10C3 = \binom{10}{3}$.

Or we could take our 10 spots and choose where to place the 7 ★'s.

This would give us $10C7 = \binom{10}{7}$.

TRY IT YOURSELF!

Find $\binom{10}{3}$ and $\binom{10}{7}$. What do you notice? Does your answer surprise you?

Remember if you have ***r*** identical objects (like ★'s) that you need to distribute into ***n*** labelled bins, then the bars (|) represent the dividers then you will always have ***n*** - 1 dividers. And the number of ways to do this is given by

$$\binom{n - 1 + r}{r}$$

If you need a refresher, check out our last newsletter activity,
Combinations without Repetition.

CHALLENGE PROBLEMS

1. A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when Shirley enters the shop, how many ways can she select a dozen donuts?

2. President Rowyn has four vice-presidents: (1) Rigley, (2) Iliana, (3) Scout, and (4) Brianna. She wishes to distribute among them $1000 in Christmas bonus checks, where each check will be written in a multiple of $100.

A) How many ways can Rowyn distribute the checks if it is possible for one or more of the vice-presidents to receive no check?

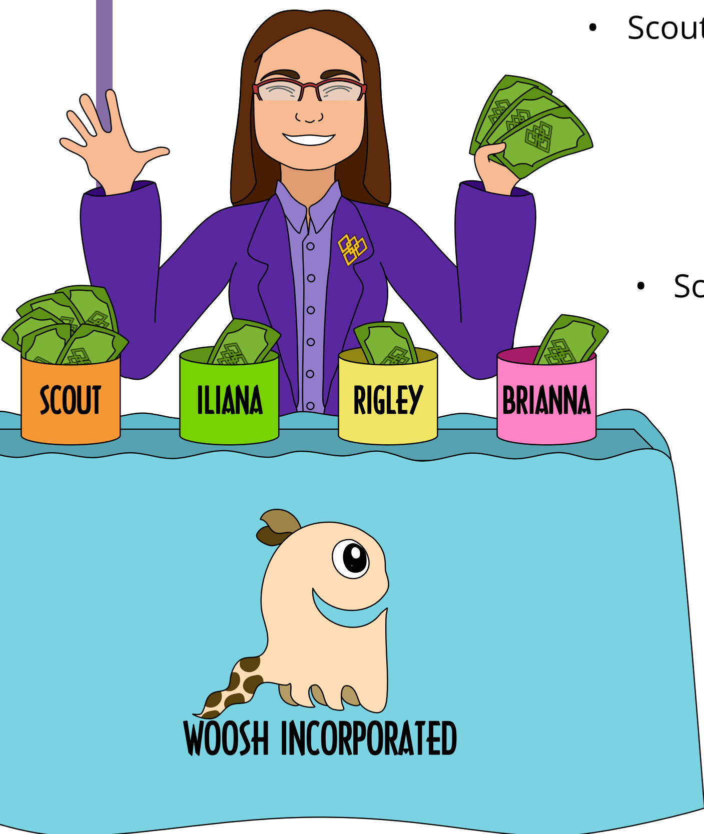
B) To avoid hard feelings, each vice-president should receive at least **$100**.

i) What is ***just one*** possible way that President Rowyn could distribute the bonus checks with this restriction?

ii) How many ***total*** ways can President Rowyn distribute the bonus checks with this restriction?

C) If each vice-president must get at least **$100** and Scout as executive vice-president gets at least **$500,** with these restrictions:

i) What is one possible way that President Rowyn could distribute the checks if:



- Scout receives exactly **$500**

- Scout receives exactly **$600**

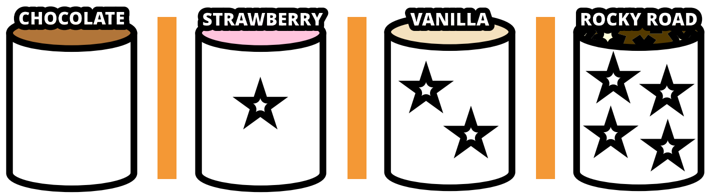
- Scout receives exactly **$700**

ii) Is it possible for President Rowyn to satisfy the restrictions and for vice-president Scout to receive exactly **$800**? Why or why not?

iii) How many ***total*** ways can President Rowyn distribute the bonus checks?

TRY IT YOURSELF SOLUTIONS

Page 2.



Rewritten:

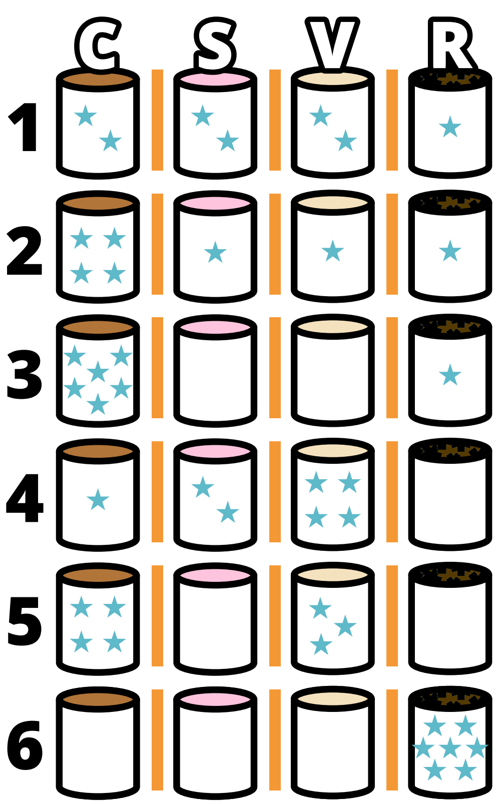
| ★ | ★ ★ | ★ ★ ★ ★

Page 4.

$$\binom{10}{3} = \frac{10!}{(10-3)!\cdot 3!} = \frac{10!}{7!\cdot 3!} = \frac{\cancel{10}\cdot\cancel{9}\cdot 8\cdot\cancel{7}!}{\cancel{7}!\,(\cancel{3}\cdot\cancel{2}\cdot 1)} = \frac{5\cdot 3\cdot 8}{1} = 120$$

$$\binom{10}{7} = \frac{10!}{(10-7)!\cdot 7!} = \frac{10!}{3!\cdot 7!} = \frac{\cancel{10}\cdot\cancel{9}\cdot 8\cdot\cancel{7}!}{(\cancel{3}\cdot\cancel{2}\cdot 1)\,\cancel{7}!} = \frac{5\cdot 3\cdot 8}{1} = 120$$

PRACTICE ACTIVITY ANSWERS



6 | | | ★ ★ ★ ★ ★ ★ ★

5 ★ ★ ★ ★ | | ★ ★ ★ |

1 ★ ★ | ★ ★ | ★ ★ | ★

2 ★ ★ ★ ★ | ★ | ★ | ★

4 ★ | ★ ★ | ★ ★ ★ ★ |

3 ★ ★ ★ ★ ★ ★ | | | ★

CHALLENGE PROBLEM SOLUTIONS

1. $\binom{20 + 12 - 1}{12} = \binom{\mathbf{31}}{\mathbf{12}}$ = 141,120,525 ways.

2A. President Rowyn is making a selection of size 10 (one for each unit of $100) from each Vice President can be represented by a bin and we have 10 that represent the 10 units of $100. This can be done in

$$\binom{4 + 10 - 1}{10} = \binom{\mathbf{13}}{\mathbf{10}} = 286 \text{ ways.}$$

2B)

i)

Rigley	Iliana	Scout	Brianna
$100	$100	$700	$100
$200	$200	$100	$500
$300	$300	$300	$100

Just some potential answers.

Your answer works as long as **the sum of each row equals $1000** and **each Vice President receives at least $100.**

ii) If each Vice President is represented by a bin and $100 units are represented by a star, and each Vice President must receive at least 1 star ($100), we place 1 of the 10 stars in each bin. This leaves 6 stars left to distribute any way that we want.

This can be done in $\binom{4 + 6 - 1}{6} = \binom{\mathbf{9}}{\mathbf{6}}$ = **84 ways.**

C) i) **If Scout receives exactly $500, all of the following are possible:**

Iliana ($100), Rigley ($100), Scout ($500), Brianna ($300)

Iliana ($100), Rigley ($200), Scout ($500), Brianna ($200)

Iliana ($100), Rigley ($300), Scout ($500), Brianna ($100)

Iliana ($200), Rigley ($100), Scout ($500), Brianna ($200)

Iliana ($200), Rigley ($200), Scout ($500), Brianna ($100)

Iliana ($300), Rigley ($100), Scout ($500), Brianna ($100)

If Scout receives exactly \$600, all of the following are possible:

Iliana (\$100), Rigley (\$100), Scout (\$600), Brianna (\$200)

Iliana (\$100), Rigley (\$200), Scout (\$600), Brianna (\$100)

Iliana (\$200), Rigley (\$100), Scout (\$600), Brianna (\$100)

If Scout receives exactly \$700, only the following is possible:

Iliana (\$100), Rigley (\$100), Scout (\$700), Brianna (\$100)

ii) No. If President Rowyn gives \$800 to Scout then she has only \$200 left to distribute to the three remaining Vice Presidents in \$100 increments.

iii) You can look at this as:

the number of ways Scout gets exactly \$500		**the number of ways Scout gets exactly \$600**		**the number of ways Scout gets exactly \$700**
n = 3 , r = 0		**n = 3 , r = 1**		**n = 3 , r = 0**
$\binom{3+2-1}{2}$	+	$\binom{3+1-1}{1}$	+	$\binom{3+0-1}{0}$
$\binom{4}{2}$	+	$\binom{3}{1}$	+	$\binom{2}{0}$
6	+	**3**	+	**1**

$$= \textbf{10 ways.}$$

Another way to solve this problem would be place 5 stars in Scout's bin and one star in each of the other Vice Presidents' bins leaving 10 - (5 + 3) = 2 stars to distribute in any four Vice President Bins.

$$\binom{4+2-1}{2} = \binom{5}{2} = \textbf{10 ways.}$$