GAIM 2017

SOLUTIONS

PLANT

Karura Nursery

Question: Mercy is 31 years old. Her daughter Faith is 7 years old. In how many years will Mercy be exactly twice as old as her daughter?

Solution 1: In N years from now, Mercy will be 31 + N years old, and Faith will be 7 + N years old. If Mercy is twice as old as Faith, we have the following equation to solve:

$$31 + N = 2 * (7 + N)$$

We can simplify this equation to:

$$31 + N = 14 + 2N$$

If we solve for N (by subtracting N and 14 from both sides), we get N = 17. Therefore, Mercy will be twice as old as Faith in 17 years.

Solution 2: Seven years ago, Mercy was 24 years old, and Faith was just born. So, 24 years after Faith was born, Mercy will be twice Faith's age. Mercy will be 48 and Faith will be 24. 24 years after 7 years ago, is 17 years from now (24 – 7=17).

Green Belt Movement

Question: Talai the Tailor is making green belts using a length of cloth measuring 58 yards. What is the smallest number of cuts she needs to make to cut the cloth into a combination of 3-yard and 7-yard lengths, with no cloth left over? (Folding and cutting multiple layers of cloth at once is not allowed)

Solution: Notice that each cut will increase the number of pieces by one. Since we start with one piece, the number of cuts will be one fewer than the number of pieces. To make the number of pieces as small as possible, we want to have as few of the shorter (3-yd.) pieces as we can.

To find the fewest number of 3-yard pieces possible, we keep subtracting 3 from 58 until the result is a multiple of 7 (and thus can be broken evenly into 7-yard lengths). We quickly deduce that the smallest number of 3-yard pieces is three, which leaves 49 yards to be divided into 7-yard lengths. Therefore, the most optimal way to cut the cloth is into three 3-yard lengths and seven 7-yard lengths (10 pieces total), which can be accomplished in 9 cuts.

I Only Speak

Question: Each of the 9 nurses in a Kenyan hospital speaks exactly two languages. Three nurses speak only Swahili and English, three speak only Kikuyu and English, and three speak only Swahili and Kikuyu. In how many ways can Dr. Chekorir choose a pair of nurses who, together, can speak all three languages?

Solution 1: Let us first consider what kind of pairs of nurses will work:

If both nurses come from the same group, we are out of luck; the two nurses will speak the same pair of languages, and nobody will be there to speak the third one.

On the other hand, any two nurses from different groups will speak all three languages between them.

Therefore, we can choose the first nurse randomly (9 choices) and choose the second nurse from one of the two other groups (6 choices), giving us a total of 9*6 = 54 choices. However, we counted each possible pair twice. For example, if a pair is made up of Nurse Anna and Nurse Beth, we counted the pair once when we chose Anna as the first nurse and Beth as the second one, and then again when we chose Beth as the first nurse and Anna as the second nurse. Each pair was counted twice, so 54 is twice the number of possible pairs. Therefore, the number of possible pairs is 54 ÷ 2 = 27.

Solution 2:

The pair can be

- A nurse that speaks Swahili and English (S&E) with a nurse that speaks Kikuyu and English (K&E). For that there are 9 choices (3 choices for S&E nurse * 3 choices for K&E nurse).
- Another possibility is an S&E nurse with a nurse that speaks Swahili and Kikuyu (S&K). In this case, there are also 9 also choices (3 choices for S&E nurse * 3 choices for S&K nurse).
- Finally, we can have a K&E nurse and an S&K nurse. Again, there are 9 (3 choices for K&E nurse * 3 choices for S&K nurse) possibilities.

Those are the only choices of different groups (again, two nurses from the same group will not work), so there are 9 + 9 + 9 = 27 pairs.

CLIMB

Fast or Slow?

Question: Had Junko climbed Mt Fuji twice as fast as she did, she would have reached the top in two hours. How many hours would it have taken her, had she climbed twice as slow as she did?

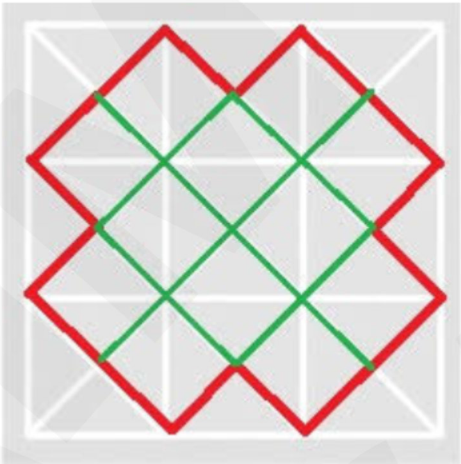
Solution: Since it would have taken Junko 2 hours, if she were twice as fast, it took twice that time, or 4 hours. Had she gone two times slower than that, it would have taken her twice as long, or 8 hours.

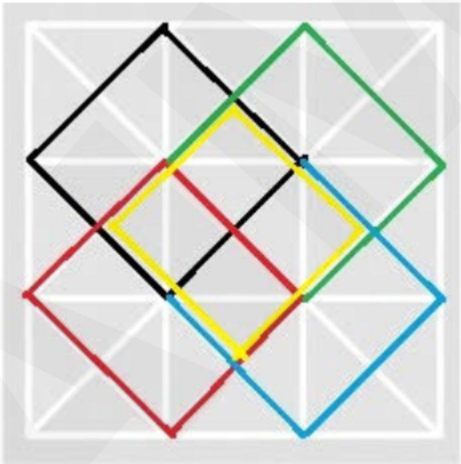
Base Camp

Question: How many squares of any size can be seen on the flag above?

Solution: Let us first consider squares whose sides are horizontal and vertical. Those are of size 1x1, 2x2, and 3x3. There are 9 (3*3) 1x1 squares, 4 (2*2) 2x2 squares, and 1 3x3 square, for a total of 14 squares.

There are also squares whose sides go sideways. These come in two sizes. The small ones (see the figure on the left) form a sideways + (plus) shape (shown in red); there are 2, 4, 4, and 2 of these (going diagonally). There are also sideways squares that are twice the size of these and there are 5 of them (see the figure on the right); 4 of them form the edges of the + shape and the 5th one is in the middle of said + shape (shown in yellow).





That are a total of 17 (12+5) sideways squares and a grand total of 31 squares.

Vitamins

Question: Junko eats three vitamins every morning. She always eats yellow vitamins before orange ones, and orange vitamins before red ones. She can eat 0, 1, 2, or 3 vitamins of each color.

If the first vitamin Junko ate this morning was orange, in how many ways could she have eaten her vitamins today?

Solution: Since the first vitamin is orange, the second one can either be orange or red. If the second vitamin is red, then the last vitamin must also be red. On the other hand, if the second vitamin is orange, the last vitamin can be either orange or red. Therefore, there are a total of 3 possibilities:

Orange–Red–Red

Orange–Orange–Orange

Orange–Orange–Red

CODE

The Biggest Difference

Question: Rosalind erases the same number of digits from each of the following two numbers: 123456789 – 987654321.

She then takes the difference.

What is the biggest number she can obtain?

Solution: First of all, note that since we delete the same number of digits from each number, the resulting two numbers will also have the same number of digits.

If we delete 3 or fewer digits, the result will be negative. In order for the two numbers to have the same number of digits in this scenario, the largest that the first digit of the first number can be is 4, while the smallest that the first digit of the second number can be is 6.

If we delete 4 digits, the largest the first number can be is 56,789 and the smallest the second number can be is 54,321; this gives a maximum possible difference of 2,468.

Similarly, if we delete 5 digits, the largest the first number can be is 6,789 and the smallest the second number can be is 4,321; this gives the same difference of 2,468.

If we delete 6 or more digits, the two numbers that we are subtracting will have 3 or fewer digits and so the result will have 3 or fewer digits and so it will be smaller than 2,468.

Therefore, the biggest number that Rosalind can obtain is 2,468.

Secret Lab

Question: The code to open a secret lab is a 2-digit prime number. We guessed 79 and 13. Both guesses contained one correct digit in the wrong place.

We have one more guess to open the door. What is the code?

Solution: From the first attempt we know that the code is either 9_ or _7 (where _ represents some digit). From the second attempt we know that the code is either 3_ or _1. We can combine this information in only 2 ways:

- 9_ with _1 yields 91. However, 91 is not a prime (91 = 13 * 7)

- 3_ with _7 yields 37, which is a prime.

Therefore, the code is 37.

Write in Numbers

Question: Rosalind writes in a secret language. She takes each letter's place in the alphabet (A=1, B = 2, …, Z = 26) and

- Writes that number if the number is even
- Writes 26 minus that number if the number is odd

Which 6-letter English word is 72018111419?

Solution: Each letter is encoded by a number between 1 and 26. Reading normally (left-to-right), the first letter is either encoded by 7 or 72. Since 72 > 26, the number must be 7. 7 is odd, so it encodes a letter which is 26-7 = 19th in the alphabet, that is, S.

Now we have 10 digits left and those should map to 5 more letters. So each of the remaining letters maps to a 2-digit number (if any of them mapped to 1-digit number, some other letter would have to map to a 3 or more digit number, which is impossible).

Now that we know how to split the remaining digits into numbers that correspond to letters, we finish by simply using the rules:

20 →→20th letter = T

18 →→18th letter = R

11 →→15th letter = O

14 →→14th letter = N

19 →→7th letter = G.

Therefore, the 6-letter word is: STRONG.

PARTY

18's A Dream

Question: Sophie is thinking of a three-digit number. She multiplies the digits and gets 18. How many distinct three-digit numbers could Sophie be thinking about?

Solution:

The divisors of 18 are 1, 2, 3, 6, 9, and 18. Of these, the first 5 may be used as digits.

What are the possibilities for the largest digit?

- If the largest digit equals 9, then the other two digits multiply to 2, which can only be factored as 2*1 (writing the largest factor first). In this case, Sophie's digits are 9, 2, and 1. There are **6** possible numbers that you can make with these digits (3 choices for the first digit, 2 for the second, and 1 for the last digit). [All 6 numbers are: 921, 912, 291, 219, 192, 129]
- If the largest digit equals 6, then the other two digits multiply to 3 which can only be factored as 3*1. With the digits 6, 3, and 1, we again have **6** possible numbers, using the same logic as above. [All 6 numbers are: 631, 613, 361, 316, 163, 136]
- If the largest digit equals 3, then the other two digits multiply to 6, which can be factored as 6*1 or as 3*2. However, the first of these would have 6 as the largest digit and not 3. Therefore, in this case, the digits must be 3, 3, and 2. There are **3** possible numbers that can be made with these digits: 332, 323 and 233.
- 2 or 1 can't be the largest digit. (The product of the digits would then be at most 2*2*2 = 8, which is less than 18).

Adding up the number of distinct possibilities in the 3 cases, we get 6 + 6 + 3 = 15 numbers that Sophie may be thinking about.

Juggle 2017

Question: How many positive numbers can you make with the numbers 2, 0, 1, 7 separated by "+" or "–" signs? (For example, 7 – 0 + 2 + 1 = 10)

Solution 1: The sign in front of 0 does not matter. Also, the order of the numbers does not matter, as long as we move each number together with its sign.

The number 7 must come with a + (plus) sign, otherwise the result will be negative (–7+1+2 < 0) Other choices of signs in front of 1 and 2 give a smaller result, but if 7 comes with a plus sign, any choice of signs in front of 1 and 2 will give us positive number (Again, because 7–2–1>0).

If we evaluate those 4 choices (7+1+2, 7+1–2, 7–1+2, and 7–1–2), we get different results (10, 6, 8, and 4). Therefore, 4 positive numbers can be obtained.

Solution 2: Note that, if we flip some signs in the expression 0 + 1 + 2 + 7 from pluses to minuses, the **parity** of the answer does not change. Also, note that the largest number occurs when all the signs are +'s, which leads to the answer of 0 + 1 + 2 + 7 = 10. It is pretty easy to see by inspection that 2 is not achievable and all the other even numbers between 4 and 10 are. Therefore, 4 possible numbers (4, 6, 8, and 10) can be obtained.

Pyramid

Question: What is 2017 – 2016 + 2015 – 2014 + 2013 – 2012 + … + 3 – 2 + 1 equal to?

Solution: If we group together pairs of numbers (going from left to right), leaving the number 1 at the end unpaired, each pair gives 1: (2017–2016) + (2015–2014) + … + (3–2) + 1. There are 2016/2=1,008 pairs and the leftover 1, so the result is 1008 + 1 = 1009.

DIAMOND

Color Cube

Question: Put digits 1, 2, 3, 4, 5, 6 inside squares A, B, C, D, E, F so that:

— The sum of the numbers in squares A, B, C, D is 11.

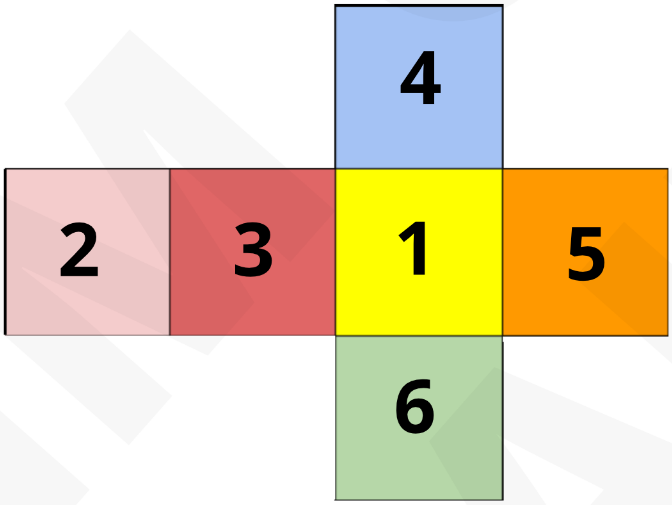
— The sum of the numbers in squares E, C, F is 11.

— The numbers in squares A, B, D increase from left to right.

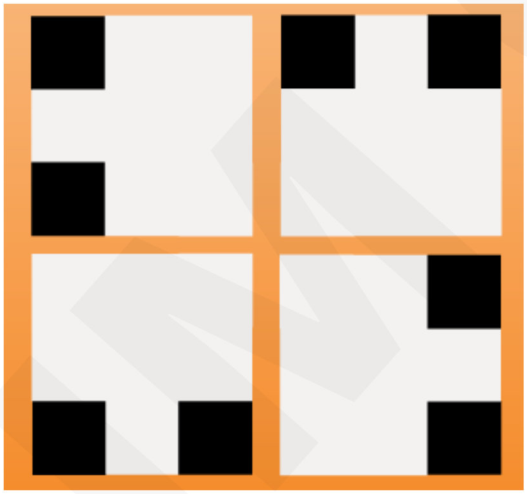
— The number in square E is smaller than the number in square F.

Solution: The first condition tells us that the sum of 4 distinct positive whole numbers (A+B+C+D) is 11. However, 1+2+3+4 =10 is the minimum possible such sum and the only way to get sum that is one higher is to replace 4 by 5, so the letters A, B, C, and D represent numbers 1, 2, 3, and 5, in some order.

That means that E and F are 4 and 6. The second condition says that E+C+F = 11, so C=1 and thus A, B, and D stand for numbers 2, 3, and 5 in some order. Since the third condition says they increase left-to-right, A=2, B=3, D=5. Finally, the 4th condition says E<F, so E=4 and F=6.



Plenty of Plates



Question:

3x3 square plates are made by joining two black and seven white 1x1 squares. How many different patterns are there?

(Remember, rotating a plate doesn't change the plate's design, such as in the figure above, but plates cannot be flipped over, as there is no design on the back)

Solution: A black square can be located in one of three locations on the plate to make a unique design. It can be located in the exact middle of the plate, in one of the plate's 4 corners, or on one of the plate's sides. There are two black squares per plate. If we understand how many of each type of plate designs are possible, we will be able count patterns and make sure that two different patterns cannot be rotated onto each other, since that would not change the plate's design.

Let us look at the possibilities (Whenever we mention two squares as a description of the pattern, we mean those two squares are black).

- The middle square is black. Then there are two patterns:
  - Middle and Corner
  - Middle and Side
- The middle square is white. Let us consider the choices for the two black squares:
  - Two corners – two patterns: opposite corners and neighboring corners
  - Two sides – again two patterns: opposite sides or neighboring sides
  - A side and a corner. We can rotate the plate so that the black side square is on the bottom. Then the corner can be any of the 4 corners, for 4 patterns.

How do we know that those 4 side-and-corner patterns are all different, that is, cannot be rotated onto each other? We already specified the rotation when we insisted that the black side square is on the bottom. Any rotation to match the position of the corners would un-match the side.

Similar ideas about rotating the plate into some specified position can be used to show that there really is only one pattern in the other cases, for example, one pattern with middle and a corner square. Therefore, there are a total of 10 patterns.

Fly, Dragon, Fly!

Question: A king is gifting each of his daughters a dragon kite. The rules:

(1) No child can receive a kite of the same number (i.e. his third child cannot receive the third kite).

(2) If a child receives an odd numbered kite, the next child cannot also receive an odd numbered kite.

(3) If a child receives a kite, the next child cannot receive the next-numbered kite. However, the next child can receive the previous-numbered kite. For example, if the fourth child receives kite #5, the fifth child cannot receive kite #6.

(4) His first child receives kite #3.

Which kite does each of his seven daughters receive?

Solution: Condition 2 tells us that two consecutive children cannot have odd-numbered kites. Thus, two odd-numbered kites must be at least two apart, and the first and the last kites must be at least 3*2=6 apart, and hence must be given to the 1st and 7th child, respectively. Therefore the pattern must be OEOEOEO.

For each child, let us represent the possible gifts by a list of their numbers and let us separate the children's data by " | ."

Condition 4 tells us that first child received Kite #3, so using what we know, the possibilities for each child so far are: 3 | 2,4,6 | 1,5,7 | 2,4,6 | 1,5,7 | 2,4,6 | 1,5,7. That is, the first child received Kite #3, and all the other children must receive a kite other than #3, following the pattern of OEOEOEO.

Applying Condition 1, where each child cannot receive a kite of the same number, makes the possibilities for each child: 3 | 4,6 | 1,5,7 | 2,6 | 1,7 | 2,4 | 1,5.

Finally, by applying Condition 3, the 2nd child cannot get Kite #4 (One higher than what the first child got). This simplifies the possibilities to: 3 | 6 | 1,5,7 | 2 | 1,7 | 4 | 1,5.

Using Condition 3 again, the third child cannot get Kite #7, so the only child who can get Kite #7 is the 5th child, making the possibilities: 3 | 6 | 1,5 | 2 | 7 | 4 | 1,5.

Finally, the last child cannot get Kite #5 (one higher than 6th child's Kite), So the solution is

3 | 6 | 5 | 2 | 7 | 4 | 1

Child 1: 3   Child 4: 2
Child 2: 6   Child 5: 7
Child 3: 5   Child 6: 4
Child 7: 1

Don't Bug Buggy!

Question: Buggy the Bug loves exercise. She starts at vertex A and moves clockwise around the vertices of a regular octagon with side length 1. Her first trip of length 3 takes her from A to D. She then takes trips of lengths 5, 13, 3, 5, 13, 3, 5, 13, 3, and so on. Which letter does she land on after her 2017th trip?

Solution: The length of the pattern in the lengths of the trips is 3, so Buggy will repeat that pattern of 3, 5, 13, etc. 672 times (2017÷3) and then do an extra trip of length 3, since 2017/3 gives a remainder of 1.

Each time Buggy travels a length of 8, she returns to the starting point (A). Therefore we only need the remainder that we get when we divide the total length of the trip by 8 to determine the endpoint.

The total length of the trip is 672 * (3 + 5 + 13) + 3 = 672 * 21 + 3.

672 is divisible by 8, so 672 * 21 is divisible by 8, and thus the remainder of (672 * 21 + 3) / 8 is 3. That means that the buggy will end up 3 points after the starting point, that is, at Letter D.