GAIM 2019

Round 1 Solutions
Middle Division

IMMORTALS: Storm

QUESTION

Zhayedan ran three miles in a straight line. She ran the first mile at a speed of 4 miles per hour. She ran the next mile at a speed of 8 miles per hour.

How fast did she run the third mile, in miles per hour, if her average speed was 6 miles per hour?

ANSWER

8 miles per hour.

SOLUTION

To average 6 miles per hour (mph) for the entire 3-mile run, Zhayedan must run the entire 3 mile distance in:

$$\frac{3\ miles}{(6\ miles/hour)} = \frac{1}{2}\ hour$$

Running the first mile at 4 mph took:

$$\frac{1\ miles}{(4\ miles/hour)} = \frac{1}{4}\ hour$$

She ran the second mile twice as fast, so it took half as long:

$$\frac{1\ miles}{(8\ miles/hour)} = \frac{1}{8}\ hour$$

So to achieve the 6 mph target, the last mile must take

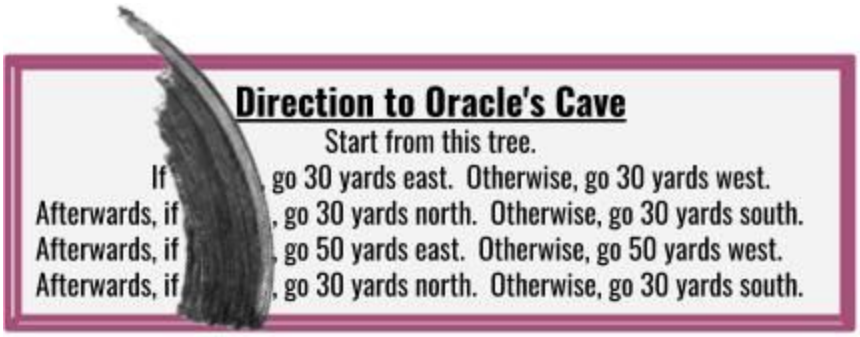
$$\frac{1}{2}\ hour - \frac{1}{4}\ hour - \frac{1}{8}\ hour = \frac{1}{8}\ hour$$

This is the same time it took to run the second mile, so it should be run at the same pace: 8 mph.

IMMORTALS: Oracle

QUESTION

Gharghâvol the Clumsy Bird erases either “10 is prime” or “10 is composite” from each line of directions to Yolo the Oracle’s Cave.



The directions now read:

(1) Start from this tree.
(2) If <smudged>, go 30 yards east. Otherwise, go 30 yards west.
(3) Afterwards, if <smudged>, go 30 yards north. Otherwise, go 30 yards south.
(4) Afterwards, if <smudged>, go 50 yards east. Otherwise, go 50 yards west.
(5) Afterwards, if <smudged>, go 30 yards north. Otherwise, go 30 yards south.

The directions lead to Yolo’s Cave. How many different possible locations are there for Yolo’s Cave?

ANSWER

12 possible locations.

SOLUTION

By following directions (2) and (4), Gharghâvol can move East-West in one of four possible ways:

30E + 50E = 80 East

30E + 50W = 20 West
30W + 50E = 20 East
30W + 50W = 80 West

Similarly, by following directions (3) and (5), Gharghâvol can move North-South in one of three possible ways:

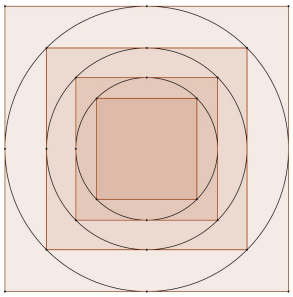
30N + 30N = 60 North
30N + 30S = 0 North
30S + 30N = 0 North *(Note that this is the same as in the line above)*
30S + 30S = 60 South

For each of the four possible East-West moves, there are three unique North-South moves possible. Therefore there are 4 x 3 = 12 possible final locations.

IMMORTALS: YOLO

QUESTION

If the smallest square has an area of 1 square foot, what is the area of the largest square in the diagram above?



ANSWER

8 square feet.

SOLUTION #1

We are told that the area of the smallest square is 1 square foot.

Now let's figure out the area of the second smallest square. If we look at the two smallest squares and the circle between them, we really only know two things:

(1) the area of the smallest square is 1 and

(2) the diameter of the smallest circle is equal to both the diagonal of the smallest square and the side length of the second smallest circle.

(1) tells us that the side length smallest square is 1 foot. Using Pythagorean theorem, we can find the length of its diagonal, which is $\sqrt{2}$ feet.

(2) tells us that side of the second smallest square is equal to the diagonal of the smallest, which we now know is $\sqrt{2}$ feet.

So the area of the second smallest square is $\sqrt{2} * \sqrt{2} = 2$ feet.

This is double the area of the smallest square. Similarly, each subsequent square is double the area of the prior square. With 4 squares, we can see their areas are 1, 2, 4, and 8 square feet.

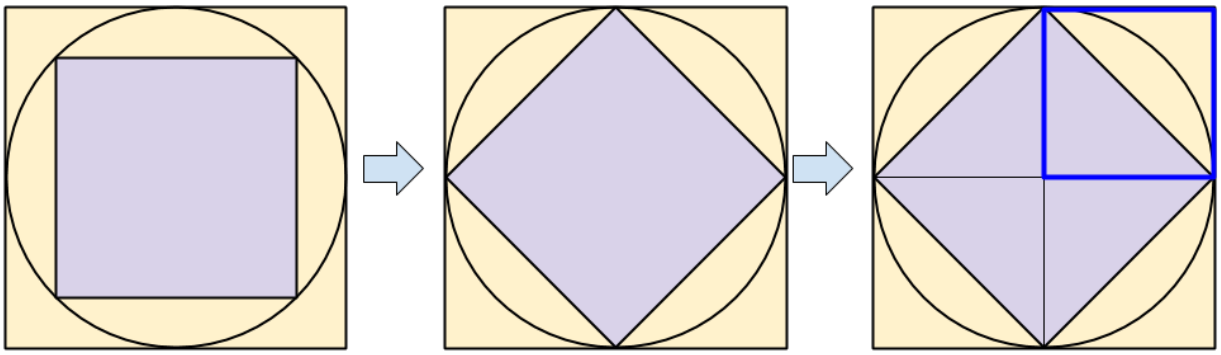
So the answer is 8 square feet.

SOLUTION #2

Another way to start this problem is to, again, focus on the two smallest squares and the circle connecting them.

First, we rotate the smallest square of side length 1 by 45°. (See the middle diagram below.) This does not change the area of any of our objects.

Now, draw the diagonals of the smaller square. In each quadrant, the purple triangle is half the size of the area outlined in blue, and therefore the purple square is half the size of the larger yellow square.



This argument will hold for each quadrant, so each square is twice the size of the square nested inside of it. In order of size, from smaller to larger, the areas of the four squares are 1, 2, 4, and 8 square feet.

So the answer is 8 square feet.

FREEDOM: Mbundu

QUESTION

Garina-Mbiji is selling fish at the market at a listed price of $2.50 per pound. Nzinga chooses a big blowfish weighing 3.2 pounds. Garina-Mbiji cheerfully offers it to Nzinga for $6. What is the percentage discount from the listed price that Nzinga gets?

ANSWER

25% discount.

SOLUTION

The listedlisted price of the fish isis3.2lbs x $2.50/lb = $8.00

Note: = $8.00.

$6 is a $2 discount from the full price. As a percent, $2 is 25% of $8, so the answer is 25%.

Note: We can take an arithmetic shortcut by moving a factor of 4 from the 3.2 to the 2.5, as follows:

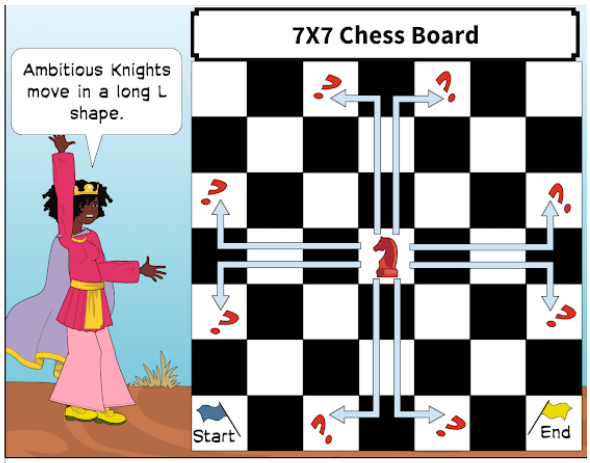
$$
\begin{aligned}
3.2lbs \times \$2.50\ /lb &= 3.2lbs \times (1) \times \$2.50/lb \\
&= 3.2lbs \times (\tfrac{1}{4}\times 4) \times \$2.50/lb \\
&= (3.2lbs \times \tfrac{1}{4}) \times (4 \times \$2.50/lb) \\
&= (0.8lbs) \times (\$10/lb) \\
&= \$8
\end{aligned}
$$

FREEDOM: Knights

QUESTION

In one turn, an “ambitious knight” moves three squares in one direction and one square in an orthogonal direction (as opposed to a regular chess knight which moves by 2 squares in one direction and 1 square in an orthogonal direction).

What is the smallest number of turns it takes to move an ambitious knight from one corner to a neighboring corner of an 11x11 chessboard?



ANSWER

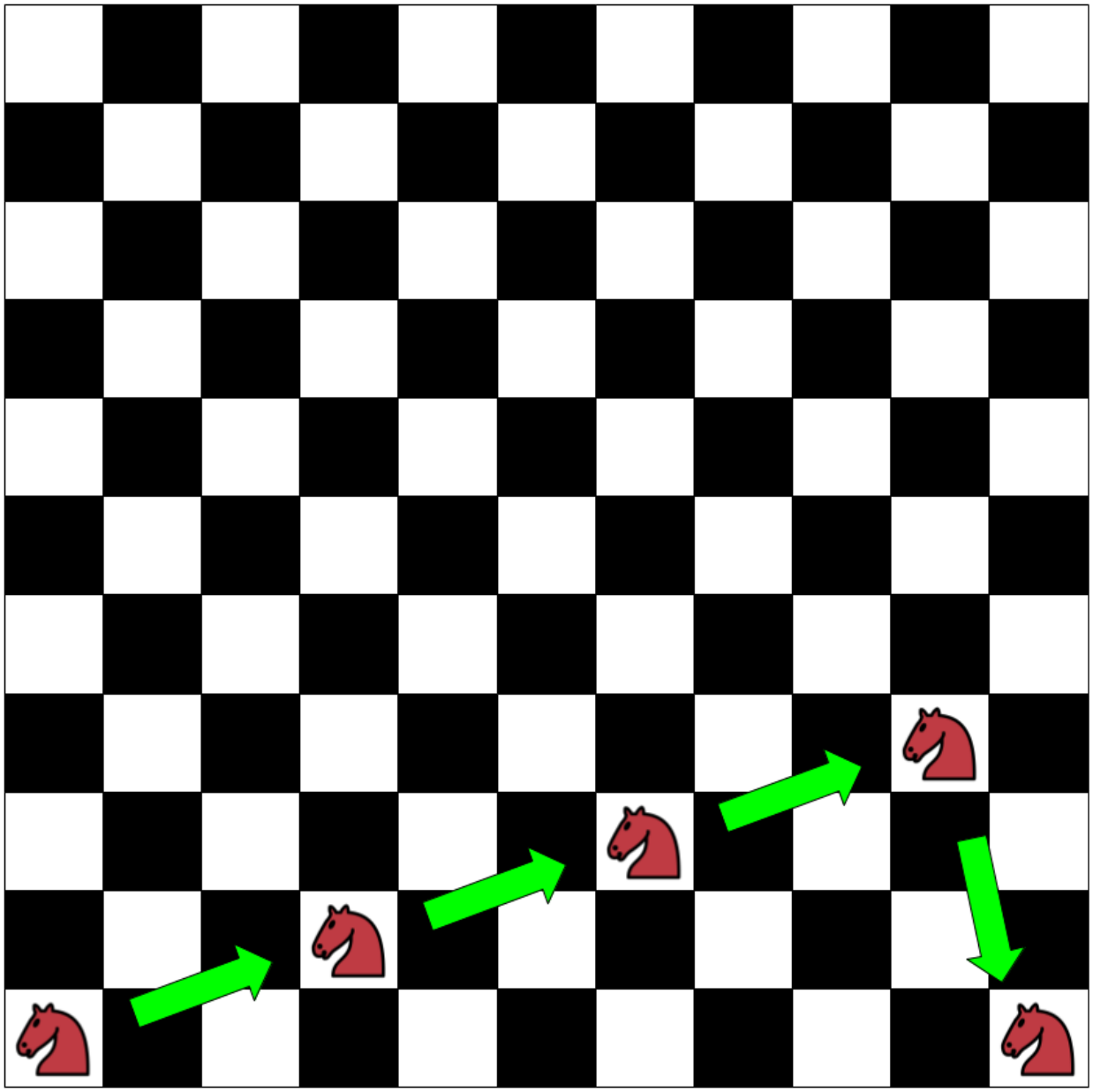
4 moves.

SOLUTION

We start in the lower left corner.

First, make 3 moves that are 3 to the right and 1 up. Then make 1 move that is 1 to the right and 3 down.

The 3 ups in the first three moves will cancel with the 3 down on the last move and there will be a total of 10 moves to the right -- just the number to go from one corner to the next.



FREEDOM: Galore

QUESTION

Fill in the crossword puzzle using each digit from 1 to 9 exactly once.

Across:

1. A number consisting of three consecutive digits, either in an increasing or in a decreasing order.
2. A number consisting of three consecutive digits, either in an increasing or in a decreasing order.
3. A number consisting of three consecutive digits, either in an increasing or in a decreasing order.

Down:

1. A perfect square.
2. A composite number.
3. A composite number.

ANSWER

1 1	2 2	3 3
4 9	8	7
5 6	5	4

SOLUTION

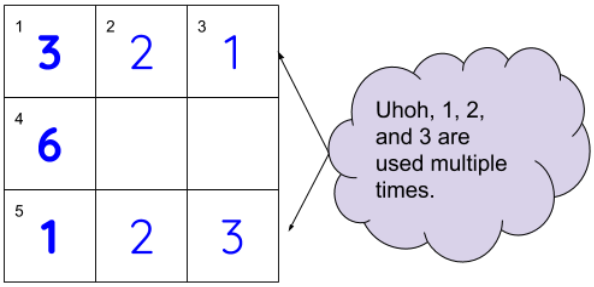
The three across clues tell us that the three numbers, when read across, must be formed by consecutive digits. That means that the three numbers must be 123, 321, 456, 654, 789 or 987.

(1 Down) is a three-digit number that is a perfect square. The three digit perfect squares are: 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900 and 961.

Since (1 Down) must start with the same digit as (1 Across), its first digit must be 1, 3, 4, 6, 7, or 9. We can first eliminate all the possibilities that have a 2, 5, 8 or 0 in them. This leaves 144, 169, 196, 361, 441, 676, and 961.

We can also eliminate perfect squares with duplicate digits, kicking out 144, 441, 676. We now have 169, 196, 361, and 961 as potential options for (1 Down).

Finally, we can eliminate 361 as that would require the 321 for (1 Across) and 123 for (5 Across) which violates using each digit exactly once.



Let's try 169, 196, and 961.

Each of these satisfies the clues for 1 Down and all the Across clues. We can quickly see they also satisfy the 2 Down clue as 258, 285, and 852 are all composite. We see that (2) satisfies the 3 Down clue as 374 is even and, hence, composite. As 347 and 743 are both prime, (2) is the only solution.

1 **1**	2 2	3 3
4 **6**	5	4
5 **9**	8	7

Across:

1. 123 has consecutive digits
2. 654 has consecutive digits
3. 987 has consecutive digits

Down:

1. 169 is a perfect square.
2. 258 is a composite number.
3. **347 is prime** 😞

1 **1**	2 2	3 3
4 **9**	8	7
5 **6**	5	4

Across:

1. 123 has consecutive digits
2. 987 has consecutive digits
3. 654 has consecutive digits

Down:

1. 196 is a perfect square.
2. 285 is a composite number.
3. 374 is a composite number.

This combination works! 😎

1 **9**	2 8	3 7
4 **6**	5	4
5 **1**	2	3

Across:

1. 987 has consecutive digits
2. 654 has consecutive digits
3. 123 has consecutive digits

Down:

1. 961 is a perfect square.
2. 852 is a composite number.
3. **743 is prime.** 😟

RADIUM: Joke's On Them

QUESTION

Marie is thinking of a positive integer. It is divisible by 11. When she crosses out all repeating digits, she gets 8. What is the smallest number she could be thinking of?

ANSWER

484.

SOLUTION

Could the number have just one digit?

8 is the only one-digit number with an 8 in it but it is not divisible by 11, so no.

How about 2 digits?

Well, if it has two digits and it loses one of them when eliminating repeating digits, then both must be eliminated and we can't end up with 8.

How about 3 digits?

If so, it'd be of the form ##8, #8#, or 8## for some digit # other than 8. It can't be of the first form since ##0 is divisible by 11 for all positive digits #, so 8 more than ##8 cannot be divisible by 11.

Similarly, 8## is never divisible by 11 because ## is divisible by 11 and 800 is not.

So we need only consider numbers of the form #8#. There is a divisibility rule for 11, so let's use that. It says that #8# is divisible by 11 if and only if x - 8 + x is divisible by 11. (Note this sum might be negative or zero.)

With the restriction that # is a digit, we can see that # = 4 is the only valid solution, therefore the answer is 484.

RADIUM: Poison

QUESTION

Potion A is poison. Potion B is water. Potion C is lemonade. Potion D is antidote. There is an unlimited supply of each potion.

Shirley takes three sips in a row, not necessarily of different potions. If she takes at least one sip of the poison, she will be poisoned unless she also drinks the antidote. If Shirley drinks a sip of antidote at any time, it will protect her from even a double dose of poison. How many different drink orders result in her not being poisoned at the end of her three sips?

(For example, antidote, poison, poison, and poison, poison, antidote are both safe drink orders.)

ANSWER

45.

SOLUTION #1

Let's divide our counting of safe orderings into 2 groups.

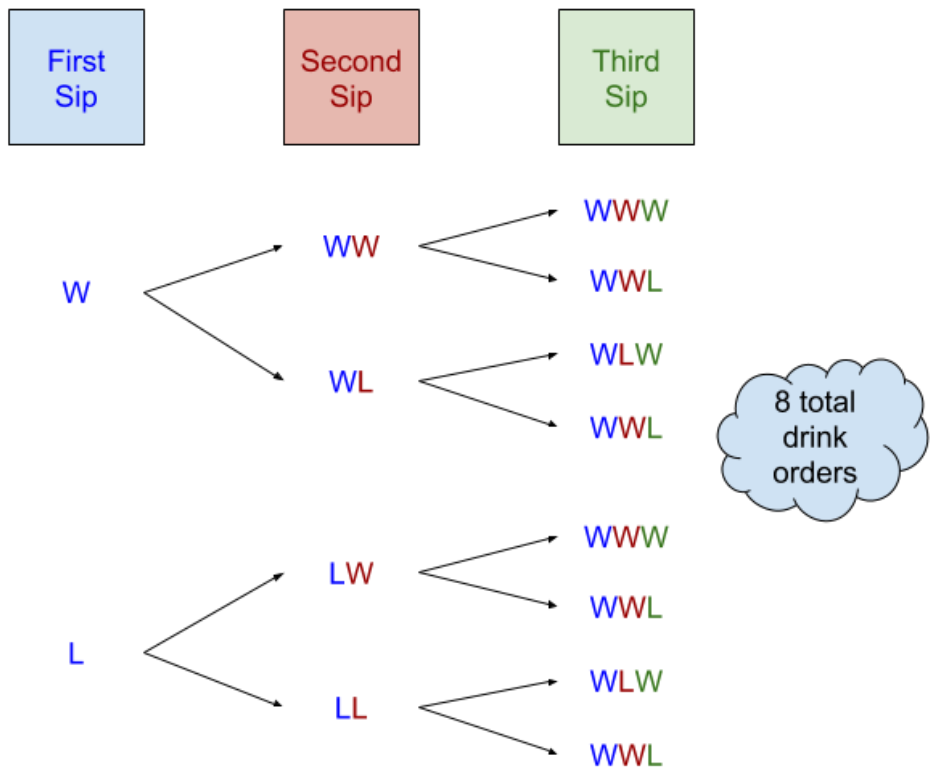
Group 1: Orders that do not include that antidote

Group 2: Orders that include the antidote.

<u>GROUP 1:</u>

For a drink series to be safe and not include any antidote, it must also not include any poison, and only include water and/or lemonade.

Shirley has two choices for each of her three sips, which equals 2 * 2 * 2 = 8 drink orderings.



<u>GROUP 2:</u>

For all drink orders that include an antidote, we separate the orderings into three categories, based on where the first antidote is.

Category 1: FIRST SIP IS ANTIDOTE
If Shirley drinks an antidote on the first sip, she can drink any of the four drinks in her next two sips. This is 1 * 4 * 4 = 16 orderings.

Category 2: FIRST SIP IS NOT ANTIDOTE + SECOND SIP IS ANTIDOTE
With no antidote in the first slot and an antidote in the second slot, there can be anything other than an antidote in the first slot and anything in the last slot, so 3 * 1 * 4 = 12 orderings.

Category 3: FIRST TWO SIPS ARE NOT ANTIDOTE + THIRD SIP IS ANTIDOTE
Similarly, with no antidotes in the first or second slot and the antidote in the third slot, the first two slots can each by anything but antidote, so 3 * 3 * 1 = 9 orderings.

This gives a total number of orderings of 8 + 16 + 12 + 9 = 45. So 45.

SOLUTION #2

Notice that the only orderings that result in Shirley being poisoned are the ones that include poison but do not include antidote. This is represented by the red area with the poison flask in the diagram below.

All other areas (yellow and blue in the diagram below) are safe.

Let's count the total number of orderings and subtract off those that have poison but no antidote.

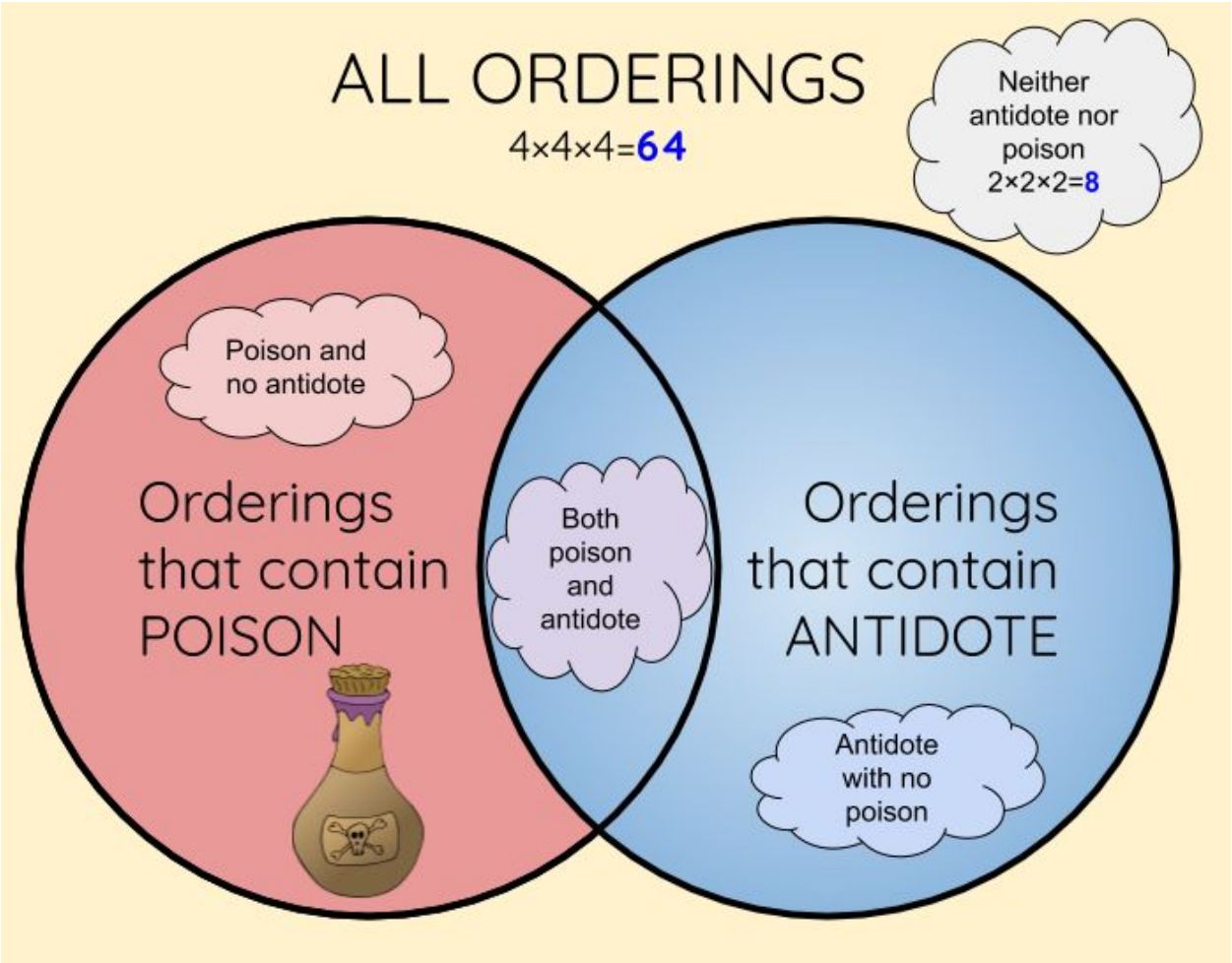
<u>How many orderings are possible?</u>
4 choices for each sip, so $4 \times 4 \times 4 = 64$.

<u>How many that have poison but do not have antidote?</u>
This is tricky.

Let's start by counting the number of orderings that don't have the antidote. This is $3 \times 3 \times 3 = 27$, and is represented by the area outside the blue circle in the Venn Diagram below.

Not all of these have poison. The orderings that only contain water and lemonade do not contain poison. This is represented by the yellow area outside of both circles. There are $2 \times 2 \times 2 = 8$ of these orderings.

Therefore, we have a total of $64 - (27 - 8) = 45$ orderings.


ALL ORDERINGS
4×4×4=64
Neither antidote nor poison 2×2×2=8
Poison and no antidote
Orderings that contain POISON
Both poison and antidote
Orderings that contain ANTIDOTE
Antidote with no poison

RADIUM: Wedge

QUESTION

An “m x n wedge” is a right triangle with legs m and n.

What is the area of the smallest square with integer sides that can be assembled with a combination of 3x4 and 5x5 wedges if both types of wedges have to be used?

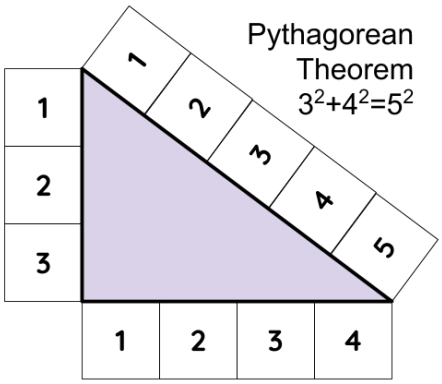
(Note: Wedges must cover the entire area of the square and may not overlap.)

ANSWER

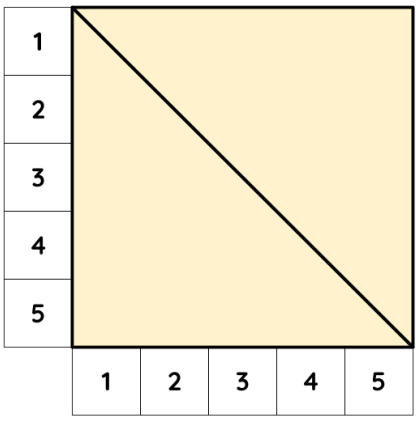
49.

SOLUTION

The first thing we notice is that a right triangle with legs of length 3 and 4 has a hypotenuse of length 5.



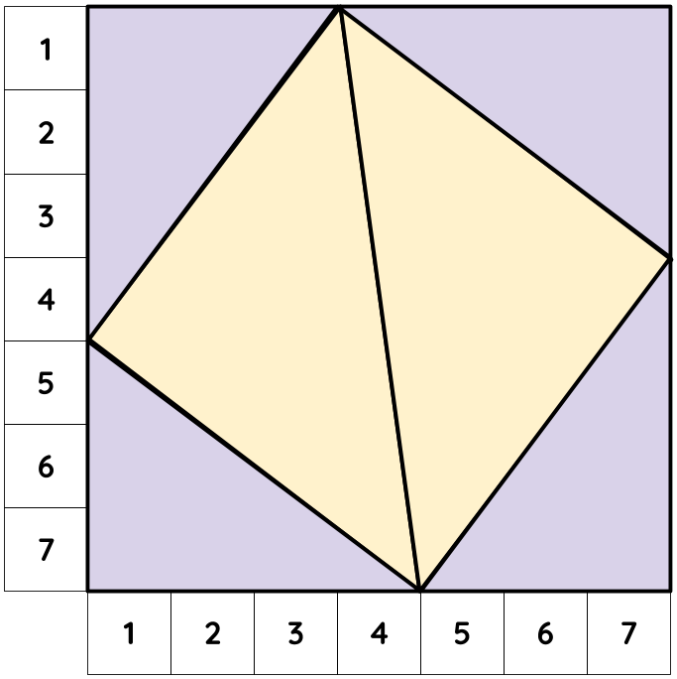
Now, we can form a square with two 5 x 5 wedges by attaching them along their diagonals.



Notice that the hypotenuse of a 3x4 wedge will line up evenly with the legs of a 5x5 wedge.

We attach four 3x4 wedges to two 5x5 wedges, as in the figure to the right.

This sides of this new square are $3+4=7$ in length, so the area is $7 \times 7 = 49$.



ORBIT: Flag

QUESTION

Rover is climbing over two huge letters, “TH” standing on a patch of flat ground. “T” has one line of symmetry, and “H” has two lines of symmetry. Each letter is 5 feet tall, 1 foot thick, and 3 feet wide. There is 1 foot between the nearest parts of “T” and “H”. If Rover travels along the edges of the two letters and on the ground, how many feet does it travel to get from the bottom left corner of “T” to the bottom right corner of “H”?

ANSWER

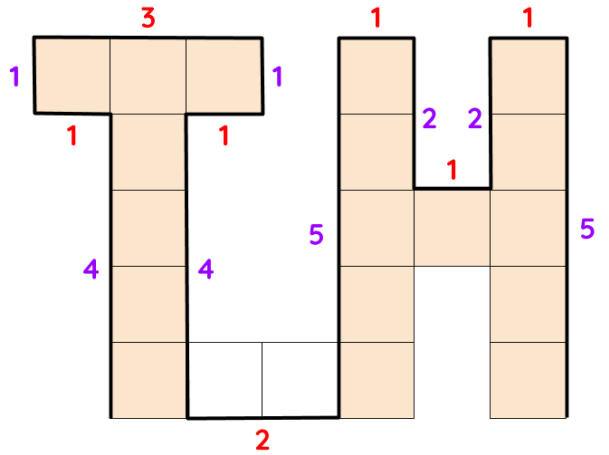
34 feet.

SOLUTION

Total vertical distance traveled: 5 (up the T) + 5 (down the T) + 5 (up the H) + .2 (down the middle of the H) + 2 (up the middle of the H) + 5 (down the H) = 24 feet.

Total horizontal distance: 1 (underside of the top of the T) + 3 (top of the T) + 1 (underside of the top of the T again) + 2 (from the T to the H) + 3 (top and middle of the H) = 10.

Adding these up, we get 34 feet.



Vertical: 4+1+1+4+5+2+2+5 = 24
Horizontal: 1+3+1+2+1+1+1 = 10

Total: 24+10 = 34

ORBIT: Jumping Jupiter

QUESTION

Words are transmitted from space to the SKYSCREEN. All words cost a certain number of Joules based on the shape of their letters. Letters that have a line of symmetry cost twice as much to transmit as letters without a line of symmetry. The word PLAY costs 18 Joules. How many Joules does the word DREAMS cost?

ANSWER

30 Joules.

SOLUTION

In PLAY, two letters -- A and Y -- have a line of symmetry. Let's say that a letter without a line of symmetry costs C Joules and a letter with a line of symmetry costs $2C$ Joules. The word PLAY will cost $C + C + 2C + 2C = 6C$ Joules.

The question tells us that the word PLAY costs 18 Joules, so C must be $18/6 = 3$ Joules. Therefore, a letter with a line of symmetry costs 6 Joules and a letter without a line of symmetry costs 3 Joules.

The word DREAMS contains four letters with a line of symmetry (D, E, A, and M), and two letters that do not. Note that S looks like it might have a line of symmetry, but it does not.

Therefore, DREAMS costs $4\times6 + 2\times3 = 30$ Joules.

ORBIT: Collision

QUESTION

The Friendship spaceship is flying from London to Madrid, then hovering in place for two hours to take some photos, and finally flying to New York.

The flight time from Madrid to New York is 8 ½ hours. The flight time from London to Madrid is 2 ½ hours.

Madrid is 1 hour ahead of London (when it is 2pm in London, it is 3pm in Madrid).
New York is 6 hours behind Madrid.

If the Friendship leaves London at 8am London time, what time will it be in New York when the spaceship arrives?

ANSWER

4pm New York time.

SOLUTION

The total time from when Friendship leaves London to when it arrives in New York is:

London to Madrid:	2.5 hours
Hover over Madrid:	2 hours
Madrid to New York:	8.5 hours
Total:	**13 hours**

If the Friendship leaves London at 8AM London time, it will arrive in New York at 9PM London time.

Now we have to convert the time zones. 9PM in London is 10PM in Madrid and 4PM in New York.

DIAMOND: Bowl Me Over

QUESTION

How many whole numbers between 1 and 100 satisfy all three of the following conditions?

(1) They have a 2 as a digit or are divisible by 2.
(2) They have a 3 as a digit or are divisible by 3.
(3) They have a 7 as a digit or are divisible by 7.

ANSWER

6 whole numbers.

SOLUTION

Let's start with the most restrictive of the 3 conditions -- the last one.

- ❏ Numbers between 1 and 100 that are divisible by 7 are 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, and 98.
- ❏ Numbers that have a digit 7 are: 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, and 97.

Now, let's eliminate the ones that don't satisfy the first condition -- that is we keep the ones neither have a 2 as a digit nor are divisible by 2.

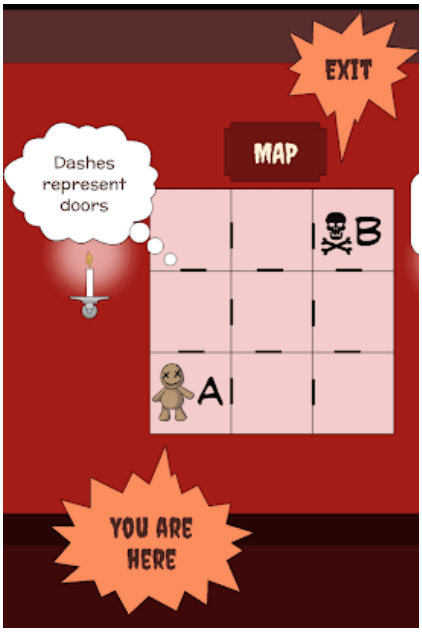
- ❏ The first set becomes 14, 21, 28, 42, 56, 70, 84, and 98.
- ❏ The second set becomes 27, 70, 72, 74, 76, and 78.

Now, eliminate ones that don't satisfy the middle condition.

- ❏ The first set becomes 21, 42, and 84.
- ❏ The second set becomes 27, 72, and 78.

So the numbers that work are 21, 27, 42, 72, 78, and 84, for a total of 6 numbers.

DIAMOND: Doom



QUESTION

Shirley is in a haunted house with nine rooms, as shown in the map to the right.

She must find her way from Room A to Room B (from the lower left corner to the upper right corner of a 3x3 square). To move from one room to another, Shirley travels through one of the twelve unlocked doors indicated on the map (inner segments of 1x1 squares).

How many distinct paths lead her from Room A to Room B if she cannot visit the same room twice?

ANSWER

12 paths.

SOLUTION

Let's label our moves as U (up), R (right), L (left), and D (down). Now, we can describe successful paths with strings like RRUU for the path that goes straight to the right and then up from there, or RRULLURR for a path that looks like the letter S.

Let's start by counting all paths that start with R. After an initial R, only a U or another R can come next. So our start looks like RU or RR. Starting with RU, there are three possibilities RUR, RUL, and RUU. Each of these leads to exactly one path: RURU, RULURR, and RUUR. If she starts with RR, a U must follow. After RRU, there are 3 routes: RRUU, RRULUR, and RRULLURR. So there are 6 routes that start with R.
Similarly, by symmetry, flipping each path across the lower-left-corner-to-upper-right-corner diagonal, there are 6 routes that start with U.

Therefore, there are 6 + 6 = 12 total paths.

DIAMOND: DIY Like A Boss

QUESTION

What is the largest number of 7 in by 8 in rectangles that can be cut from a 60 in by 60 in square?

ANSWER

64 rectangles.

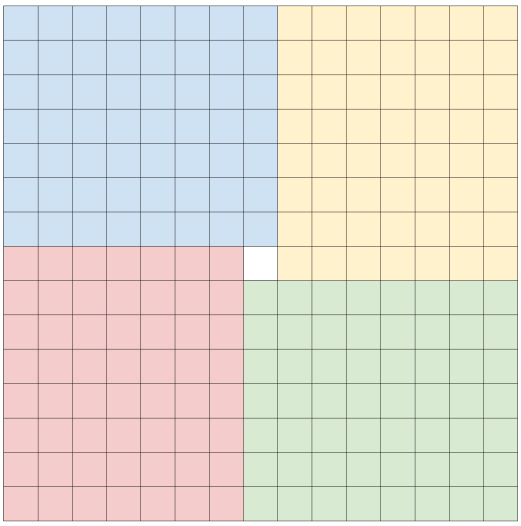
SOLUTION

We begin by cutting out a 28″x32″ rectangle from each corner so they do not overlap.

We cut each of these four 28″x32″ rectangles into four equal parts in each dimension. This gives us sixteen 7″x8″ rectangles in each of the four colored regions, or 64 7”x8” rectangles in total.

Cutting in this manner, we will only have the 4″x4″ square from the center left, which is not enough material for another 7″x8″ rectangle.

Therefore, we can cut at most 64 squares of size 7”x8” from a 60”x60” square.



Each square above measures 4”x4”

DIAMOND: The Best One

QUESTION

A team of girls is trying to crack open a secret vault. There is a grid of 100 buttons, numbered from 1 to 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The vault will open when all of the buttons are pressed. At first, each team member can press 1 button. Then, the team can press any button which is the average of any two buttons which have already been pressed.

For example, if buttons 28 and 48 have already been pressed, the team can press button 38.

What is the smallest number of team members needed to open the vault?

ANSWER

3 girls.

SOLUTION

Three girls suffice.

The basic idea is that if two buttons have been pressed, whose numbers differ by a power of two, then all buttons between the two original buttons can be pressed.

Let's look at an example. Imagine that 33 and 41 have been pressed. Their difference is $41 - 33 = 8 = 2 \times 2 \times 2$.

Next, the team can press:

- 37 (the average of 33 and 41)
- 35 (the average of 33 and 37)
- 39 (the average of 37 and 41)

Then with all the odd numbered buttons from 33 through 41, inclusive, pressed, all the even numbers in that range can be pressed.

One thing that can never happen is that the girls can never press a button lower than the lowest one they start with or higher than the highest one they start with. Taking averages never extends the range!

Therefore, the girls will need to include 1 and 100 in their initial presses, otherwise they'll never be able to reach 1 or 100. Starting with 1 and 100 alone isn't enough though because their average -- 50.5 -- isn't an integer or a button.

If the girls start with 1, 65, and 100, they will be able to reach all the buttons. Their difference is $65 - 1 = 64 = 2 \times 2 \times 2 \times 2 \times 2$. These two button presses will allow them to reach all the buttons from 1 to 65.

This will include button 36 which is 64 away from 100. With 36 and 100 pressed, the team can now work their way to pressing all of the buttons from 36 to 100.

Therefore, all the buttons can be pressed with 3 initial presses by 3 girls.